Homework 6

Analysis

Due: March 12, 2018

1. **Lebesgue Spaces:**

   (a) Prove that
$$\int_1^\infty \frac{e^{-t}}{t} \leq e^{-1}.$$

   (b) Prove the better bound:
$$\int_1^\infty \frac{e^{-t}}{t} \leq \frac{e^{-1}}{\sqrt{2}}.$$

   (c) Let $f \in L_0^p([a,b])$, $g \in L_0^q([a,b])$, $h \in L_0^r([a,b])$, where $1/p + 1/q + 1/r = 1$. Prove that
$$\int_a^b |f(t)g(t)h(t)|\, dt \leq \|f\|_{L^p}\|g\|_{L^q}\|h\|_{L^r}.$$

   (d) Let $f \in L_0^2([0,\pi])$. Is it possible to have simultaneously:
$$\int_0^\pi (f(t) - \sin(t))^2 dt \leq \frac{4}{9} \text{ and } \int_0^\pi (f(t) - \cos(t))^2 dt \leq \frac{1}{9}.$$

   (e) Suppose that $\int_0^\infty |f(t)|\, dt < \infty$. Prove or give a counterexample:
$$\lim_{t\to\infty} |f(t)| = 0.$$

2. **Energy Norms:**

   (a) f is a continuously differentiable function such that $f(0) = f(L) = 0$ and its energy norm is defined by
$$\|f\|_E = \left(\int_0^L \left(f'(x)\right)^2 dx\right)^{\frac{1}{2}}.$$
   Prove that
$$\|f\|_p \leq \left(\frac{2}{p+2}\right)^{\frac{1}{p}} (L^{1+\frac{p}{2}})^{\frac{1}{p}} \|f\|_E.$$

3. **Compactness and Equicontinuity:**

   (a) Let $f_n \in C([0,1])$ be an equicontinuous sequence of functions. If $f_n \to f$ pointwise, prove that f is continuous.

   (b) Let $K \subset C([0,1])$ be defined by
$$K = \left\{ f \in C([0,1]) : \text{Lip}(f) \leq 1 \text{ and } \int_0^1 f(x)\, dx = 0 \right\}.$$
   Prove that K is compact in $C([0,1])$ with respect to the norm $\|\cdot\|_\infty$.

4. **Differential Equations:**

(a) Consider the following scalar differential equation:

$$\frac{du}{dt} = |u(t)|^{\alpha},$$
$$u(0) = 0.$$

Show that the solution is unique if $\alpha \geq 1$, but not if $0 \leq \alpha < 1$.

(b) Suppose that $f(t,u)$ is a continuous function $f : \mathbb{R}^2 \mapsto \mathbb{R}$ such that for all $t, u, v \in \mathbb{R}$:

$$|f(t,u) - f(t,v)| \leq K|u - v|.$$

Also suppose that

$$M = \sup\{|f(t,u_0)| : |t - t_0| \leq T\}.$$

Prove that the solution of the initial value problem

$$\frac{du}{dt} = f(t,u)$$
$$u(t_0) = u_0$$

satisfies the estimate

$$|u(t) - u_0| \leq MTe^{KT}$$

for $|t - t_0| \leq T$. Explicitly check this estimate for the linear initial value problem:

$$\frac{du}{dt} = Ku$$
$$u(t_0) = u_0.$$