Homework #1

#1

Give an ε-δ proof that for all $x \in (-1,1)$:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

proof:

Let $S_K(x) = \sum_{n=0}^{K} x^n$. Since $S_K(x)$ is a geometric series it follows that

$$S_K(x) = \frac{1-x^{K+1}}{1-x}.$$

Therefore, for $x \in (-1,1)$ it follows that

$$\left| S_K(x) - \frac{1}{1-x} \right| = \left| \frac{-x^{K+1}}{1-x} \right| = \frac{|x^{K+1}|}{1-x} = \frac{|x|^{K+1}}{1-x}.$$

1. For all $\varepsilon > 0$, assume $x \neq 0$, $x \in (-1,1)$ and let

$$M(\varepsilon, x) = \frac{\ln(\varepsilon(1-x))}{\ln(|x|)} - 1.$$

Therefore, if $K > M(\varepsilon, x)$ it follows that

$$K > \frac{\ln(\varepsilon(1-x))}{\ln(|x|)} - 1.$$

Since $|x| < 1$ it follows that $\ln(|x|) < 0$ and consequently

$$\ln(|x|)(K+1) < \ln(\varepsilon(1-x)),$$

$$\Rightarrow \ln(|x|^{K+1}) < \ln(\varepsilon(1-x)).$$

Since, $\ln(x)$ is an increasing function it follows that

$$|x|^{K+1} < \varepsilon(1-x)$$

$$\Rightarrow \frac{|x|^{K+1}}{1-x} < \varepsilon.$$

2. If $x = 0$ then $S_K(x) = 1$ for all K.

By items 1 and 2 it follows that for all $x \in (-1,1)$:

$$\lim_{K \to \infty} \sum_{n=0}^{K} x^n = \frac{1}{1-x}.$$

#2.

Prove for all $x, y, z \in \mathbb{R}$,

$$|x-y| \geq ||x-z|-|z-y||.$$

proof:

Let $x, y, z \in \mathbb{R}$. Since $|x-y| = |x-z+z-y|$ it follows that

$$\begin{aligned} |x-y|^2 &= (x-z+z-y)^2 \\ &= (x-z)^2 + 2(x-z)(z-y) + (z-y)^2 \\ &\geq (x-z)^2 - 2|x-z| \cdot |z-y| + (z-y)^2 \\ &= (|x-z|-|z-y|)^2. \end{aligned}$$

Therefore, since the square root function is increasing and $\sqrt{a^2} = |a|$ it follows that

$$|x-y| \geq ||x-z|-|z-y||$$

#4.

Let a_n be a sequence in $\mathbb{R}$ satisfying for all $n \in \mathbb{N}$:

$$|a_{n+1} - a_n| < \frac{1}{n^2}.$$

Is a_n a Cauchy sequence? Prove or give a counterexample.

proof:

By repeated application of the triangle inequality it follows that for all $m, n \in \mathbb{N}$ with $m > n$ that

$$\begin{aligned} |a_m - a_n| &= |a_m - a_{m-1} + a_{m-1} - a_n| \\ &\leq |a_m - a_{m-1}| + |a_{m-1} - a_n| \\ &\leq \frac{1}{(m-1)^2} + |a_{m-1} - a_n| \\ &\leq \frac{1}{(m-1)^2} + |a_{m-1} - a_{m-2} + a_{m-2} - a_n| \\ &\leq \frac{1}{(m-1)^2} + |a_{m-1} - a_{m-2}| + |a_{m-2} - a_n| \\ &\leq \frac{1}{(m-1)^2} + \frac{1}{(m-2)^2} + |a_{m-2} - a_n| \\ &\vdots \\ &\leq \frac{1}{(m-1)^2} + \frac{1}{(m-2)^2} + \dots + \frac{1}{n^2} \end{aligned}$$

Therefore, $|a_m - a_n| \leq \sum_{k=n}^{m-1} \frac{1}{k^2} \leq \sum_{k=n}^{\infty} \frac{1}{k^2}$. Since $\sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$ it follows that

$$\lim \sum^{\infty} \frac{1}{\ } = 0.$$

#5.

Let a_n be a sequence in $\mathbb{R}$. Suppose there are numbers k, $0<k<1$ and $C>0$ such that
$$|a_{n+1}-a_n|\le Ck^n$$
for all $n\ge 1$. Is a_n a Cauchy sequence? Prove or give a counterexample.

proof:

By repeated application of the triangle inequality it follows that for all $m,n\in\mathbb{N}$ with $m>n$ that
$$\begin{aligned}|a_m-a_n| &= |a_m-a_{m-1}+a_{m-1}-a_{m-2}+\dots+a_{n+1}-a_n|\\ &\le |a_m-a_{m-1}|+|a_{m-1}-a_{m-2}|+\dots+|a_{n+1}-a_n|\\ &\le Ck^{m-1}+\dots+Ck^n\\ &\le C\sum_{i=n}^{m-1}k^i\\ &\le C\sum_{i=n}^{\infty}k^i.\end{aligned}$$
Since $0<k<1$ it follows that $\sum_{i=n}^{\infty}k^i<\infty$ and hence $\lim_{n\to\infty}\sum_{i=n}^{\infty}k^i=0$. ■

#6.

Define a sequence of real numbers x_n by $x_1=2$ and
$$x_{n+1}=\frac{x_n}{2}+\frac{1}{x_n}$$
Prove that x_n is Cauchy.

proof:

1. Calculating it follows that
$$\begin{aligned}x_{n+1}-x_n &= \frac{x_n}{2}+\frac{1}{x_n}-\frac{x_{n-1}}{2}-\frac{1}{x_{n-1}}\\ &= \frac{x_n-x_{n-1}}{2}+\frac{x_{n-1}-x_n}{x_nx_{n-1}}\\ &= \frac{1}{2}(x_n-x_{n-1})\left(1-\frac{2}{x_nx_{n-1}}\right)\end{aligned}$$

2. Let $f(x)=\frac{x}{2}+\frac{1}{x}$. Calculating, $f'(x)=\frac{1}{2}-\frac{1}{x^2}$ which on the interval $[1,2]$ has a zero at $x=\sqrt{2}$. Now,
$$f(1)=3/2,$$
$$f(2)=3/2,$$
$$f(\sqrt{2})=\sqrt{2}$$

Therefore, by the extreme value theorem it follows that $x_n \in [\sqrt{2}, 3/2]$ for all $n \geq 2$.

3. From item 2 it follows that

$$2 \leq x_n x_{n-1} \leq 9/4$$
$$\Rightarrow 8/9 \leq \frac{2}{x_n x_{n-1}} \leq 1$$
$$\Rightarrow 0 \leq 1 - \frac{2}{x_n x_{n-1}} \leq \frac{1}{9}$$

4. By items 1 and 3 it follows that

$$|x_{n+1} - x_n| = \frac{1}{2}\left|1 - \frac{2}{x_n x_{n-1}}\right| \cdot |x_n - x_{n-1}|.$$
$$\leq \frac{1}{18} |x_n - x_{n-1}|$$

Therefore, by problem #5 x_n is a Cauchy sequence. ∎