Homework #10

#1.

a.) Construct a smooth function $f_1 : \mathbb{R} \to \mathbb{R}$ satisfying:

- $\overline{\operatorname{Supp}(f_1)} = [-1,1]$, $f_1 \geq 0$.
- $\int_{-\infty}^{\infty} f_1(x)\,dx = 1$.

Solution:

Let $h(x)$ be defined by:

$$h(x) = \begin{cases} 0, & |x|>1 \\ e^{-1/(1-x^2)}, & |x| \leq 1 \end{cases}$$

Define $f_1(x)$ by:

$$f_1(x) = \frac{h(x)}{\int_{-\infty}^{\infty} h(x)\,dx}.$$

b.) Let $n \in \mathbb{N}$. Show that $f_n : \mathbb{R} \to \mathbb{R}$ defined by $f_n(x) = n f_1(nx)$ satisfies:

- $\overline{\operatorname{supp}(f_n)} = [-\frac{1}{n}, \frac{1}{n}]$,
- $\int_{-\infty}^{\infty} f_n(x)\,dx = 1$

Solution:

If $x \in \overline{\operatorname{supp}(f_n)}$, then $nx \in [-1,1]$ and thus $\overline{\operatorname{supp}(f_n)} = [-1/n, 1/n]$.
Also,

$$\int_{-\infty}^{\infty} f_n(x)\,dx = \int_{-\infty}^{\infty} n f_1(nx)\,dx = \int_{-\infty}^{\infty} f(u)\,du = 1.$$

c.) Let $g(x)$ be a smooth function. Prove that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x) g(x)\,dx = g(0).$$

proof:

$$\left| \int_{-\infty}^{\infty} f_n(x) g(x)\,dx - g(0) \right| = \left| \int_{-\infty}^{\infty} f_n(x)(g(x) - g(0))\,dx \right|$$

$$\Rightarrow \left| \int_{-\infty}^{\infty} f_n(x) g(x)\,dx - g(0) \right| \leq \int_{-1/n}^{1/n} f_n(x) |g(x) - g(0)|\,dx$$

Let $M = \max_{-1 \leq x \leq 1} |g'(x)|$. It follows from the mean value theorem that

$$\left| \int_{-\infty}^{\infty} f_n(x) g(x)\,dx - g(0) \right| \leq \int_{-1/n}^{1/n} f_n(x) \cdot M \cdot |x|\,dx$$
$$\leq \|f_n\|_\infty \int_{-1/n}^{1/n} M \cdot |x|\,dx$$
$$\leq \underline{\frac{2\|f_n\|_\infty \cdot M}{n}}.$$

Therefore,

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x) g(x)\,dx = g(0).$$

d.) Let $a, b \in \mathbb{R}$ satisfy $a < b$. Suppose $g \in C^1([a,b])$ satisfies

$$\int_a^b g(x) f(x)\,dx = 0$$

for all smooth functions $f : \mathbb{R} \to \mathbb{R}$ with compact support contained in $[a,b]$. Prove that $g = 0$.

~~proof~~:

Let $h_n(y) = f_n(x-y)$. Then,

$$0 = \int_a^b g(y) h_n(y)\,dy = \int_a^b g(y) f_n(x-y)\,dy = \int_{x-b}^{x-a} g(x-u) f_n(u)\,du$$

$$\Rightarrow g(x) = 0.$$

#2.

a.) Let $\mathbb{X}, \mathbb{Y}$ be complete normed linear spaces. Prove that a linear operator is bounded if and only if $\|L\|_{op} < \infty$.

proof:

L is bounded if and only if for all $x \in \mathbb{X}$ there exists $M > 0$ such that $\|Lx\| \leq M\|x\|_\infty$. Therefore,

$$\|L\|_{op} = \sup_x \frac{\|Lx\|}{\|x\|} \leq M.$$

b.) Let $\mathbb{X}, \mathbb{Y}$ be complete normed linear spaces. Prove that if L is a bounded linear operator then for all $x \in \mathbb{X}$,

$$\|Lx\|_{\mathbb{Y}} \leq \|L\|_{op} \cdot \|x\|_{\mathbb{X}}.$$

proof:

For all $x \in \mathbb{X}$:

$$\frac{\|Lx\|_{\mathbb{Y}}}{\|x\|_{\mathbb{X}}} \leq \sup_y \frac{\|Ly\|_{\mathbb{Y}}}{\|y\|_{\mathbb{X}}} = \|L\|_{op}$$

$$\Longrightarrow \|Lx\|_{\mathbb{Y}} \leq \|L\|_{op} \cdot \|x\|_{\mathbb{X}}.$$

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#3.

Let $\mathbb{X}, \mathbb{Y}$ be complete normed linear spaces. Prove that a linear operator $L : \mathbb{X} \to \mathbb{Y}$ is continuous at every point in its domain if and only if it is continuous at 0.

proof:

Suppose L is continuous at 0. Therefore, for all $x_n \to 0$ it follows that $Lx_n \to L0 = 0$. Now, suppose $x_n \to x$. Then,

$$Lx_n - Lx = L(x_n - x)$$

$$\Longrightarrow \lim_{n\to\infty} (Lx_n - Lx) = \lim_{n\to\infty} L(x_n - x)$$

$$= L0$$

$$= 0$$

$$\Rightarrow \lim_{n\to\infty} Lx_n = Lx.$$

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#4.

Let $1<p<\infty$ and suppose $q\in(1,\infty)$ satisfies $\frac{1}{p}+\frac{1}{q}=1$. Let $v\in L^q([0,1])$ and define $L: L^p([0,1])\to\mathbb{R}$ by

$$L(u)=\int_0^1 u(x)v(x)\,dx.$$

Prove that L is a bounded linear operator.

~~proof~~:

$$|L(u)|\le\int_0^1 |u(x)|\cdot|v(x)|\,dx$$

$$\le \|u\|_p\|v\|_q$$

$$\Longrightarrow \|L\|\le\|v\|_q.$$

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#5

Let $\delta: C([0,1])\to\mathbb{R}$ be the linear operator that evaluates a function at the origin $\delta(f)=f(0)$.

a.) If $C([0,1])$ is equipped with the norm $\|\cdot\|_\infty$ prove that δ is bounded and compute its norm.

~~proof~~:

$$|\delta f|=|f(0)|\le\sup|f(x)|$$

$$\Longrightarrow \frac{|f(0)|}{\sup|f(x)|}\le 1.$$

Now, if $f=1$ we have equality and thus $\|\delta\|=1$.

b.) If $C([0,1])$ is equipped with the norm $\|\cdot\|_{L^1}$ prove that δ is unbounded.

~~proof~~:

Let f_n be defined as in problem #1, part b. Then, $\|f_n\|_{L^1}=1$ but $|f(0)|=nf_1(0)\to\infty$. Consequently,

$$\sup_{\|f\|_1=1}|\delta f|=\infty.$$

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#6.

Define $\mathbb{K}: \mathcal{C}([0,1]) \to \mathcal{C}([0,1])$ by

$$\mathbb{K}(f(x)) = \int_0^1 k(x,y) f(y)\,dy,$$

where $k: [0,1]\times[0,1] \to \mathbb{R}^+$ is continuous. Prove that $\mathbb{K}$ is bounded and

$$\|\mathbb{K}\|_{op} = \max_{0\le x\le 1}\left\{\int_0^1 k(x,y)\,dy\right\}.$$

proof:

Suppose $\|f\|_\infty = 1$. Then,

$$\begin{aligned}|\mathbb{K}(f(x))| &= \left|\int_0^1 k(x,y) f(y)\,dy\right| \\ &\le \int_0^1 k(x,y)\cdot|f(y)|\,dy \\ &\le \int_0^1 k(x,y)\,dy\end{aligned}$$

$$\Rightarrow \|\mathbb{K}(f(x))\|_\infty \le \max_x \int_0^1 k(x,y)\,dy.$$

We obtain equality if $f=1$ and thus $\|\mathbb{K}\| = \max_x \int_0^1 k(x,y)\,dy.$ $\blacksquare$