Homework 11

Analysis

Due: April 23, 2018

1. Let X, Y be normed linear spaces. Prove that if Y is complete the $B(X,Y)$ is a complete space with respect to the operator norm.

2. **Fourier Series:** With every function $f \in C([0,1])$ we can associate a sequence a_n by
$$f(x) \mapsto a_n = \int_0^1 f(x)\sin(2\pi nx)\,dx.$$
The series a_n is called the Fourier sine series of g, and we will denote the map from $C([0,1])$ to sequences by $\mathcal{F}$.
   (a) Show that $\mathcal{F}$ is a continuous mapping between $(C([0,1]), \|\cdot\|_{L^1})$ and l^∞.
   (b) Show that $\mathcal{F}$ is a continuous mapping between $(C([0,1]), \|\cdot\|_{L^2})$ and l^2.

3. l_c is the space of all real valued sequences that have only a finite number of non-zero terms. Recall that c_0 is the space of all sequences $a_n \in \mathbb{R}$ satisfying $\lim_{n\to\infty} a_n = 0$.
   (a) If c_0 is equipped with the $\|\cdot\|_\infty$ norm, what is the dual space of c_0? That is, what is the space of bounded linear functionals?
   (b) Show that l_c is a vector space over $\mathbb{R}$.
   (c) Show that l_c is dense in the sequence space l^p with respect to the l^p norm.
   (d) Show that the closure of l_c in the sup norm is c_0.

4. Show that the mapping defined by
$$f(x) \mapsto Tf(x) = \int_0^\pi \sin(x-y)f(y)\,dy,$$
maps functions in $C([0,\pi])$ into $C([0,\pi])$. Is this mapping continuous with respect to the L^1 and L^∞ norms on $C([0,\pi])$, and if so, what are the corresponding induced norms for T?

5. Let $\mathcal{A} = \{u \in C^1([a,b]) : u(a) = \alpha \text{ and } u(b) = \beta\}$. Consider the functional $I : \mathcal{A} \mapsto \mathbb{R}$ defined by
$$I[u] = \int_a^b g(x,u)\sqrt{1 + \left(\frac{du}{dx}\right)^2}\,dx,$$
where g is smooth function.
   (a) Calculate the *weak form* of the Euler-Lagrange equations for this functional.
   (b) Calculate the *strong from* of the Euler-Lagrange equations for this functional.

6. Let $\mathcal{A} = \{u \in C^1([-1,1]) : u(-1) = 0 \text{ and } u(1) = 1\}$. Consider the functional $I : \mathcal{A} \mapsto \mathbb{R}$ defined by
$$I[u] = \int_{-1}^1 u(x)^2\left(1 - \frac{du}{dx}\right)^2\,dx.$$
Prove that I does not have a minimizer in $\mathcal{A}$.

Homework #11

#1

Let $\mathbb{X}, \mathbb{Y}$ be normed linear spaces. Prove that if $\mathbb{Y}$ is complete then $B(\mathbb{X},\mathbb{Y})$ is complete with respect to the operator norm.

proof:

1. Let $L_n \in B(\mathbb{X},\mathbb{Y})$ be Cauchy. Then, for $x \in \mathbb{X}$ it follows that
$$\|L_n(x) - L_m(x)\|_{\mathbb{Y}} = \|(L_n - L_m)(x)\|_{\mathbb{Y}} \leq \|L_n - L_m\|_{op} \cdot \|x\|_{\mathbb{X}}.$$
Therefore, $L_n(x)$ is Cauchy in $\mathbb{Y}$. Define
$$L(x) = \lim_{n\to\infty} L(x).$$

2. Since L_n is Cauchy it follows that there exists $K > 0$ such that
$$\|L_n\| \leq K.$$
Therefore, for $x \in \mathbb{X}$ with $\|x\| = 1$ it follows that
$$\|L(x)\|_{\mathbb{Y}} = \lim_{n\to\infty} \|L_n(x)\|_{\mathbb{Y}} \leq K.$$
Consequently $L \in B(\mathbb{X},\mathbb{Y})$.

3. Also, if $\|x\|_{\mathbb{X}} = 1$ it follows that
$$\|(L - L_n)x\|_{\mathbb{Y}} \leq \|(L - L_m)x\|_{\mathbb{Y}} + \|(L_m - L_n)x\|_{\mathbb{Y}}$$
There exists $N \in \mathbb{N}$ such that $m, n > N$ implies
$$\|(L_m - L_n)x\| \leq \varepsilon$$
$$\Rightarrow \|(L - L_n)x\|_{\mathbb{Y}} \leq \|(L - L_m)x\|_{\mathbb{Y}} + \varepsilon.$$
Taking $m \to \infty$ it follows that for $n > N$:
$$\|(L - L_n)x\|_{\mathbb{Y}} \leq \varepsilon.$$
$\Rightarrow L_n \to L$ in the operator norm.

#2.

With every function $f \in \mathcal{C}([0,1])$ we can associate a sequence a_n by

$$f(x) \to a_n = \int_0^1 f(x)\sin(2\pi n x)\,dx.$$

The series a_n is called the Fourier sine series of g, and we will denote the map from $\mathcal{C}([0,1])$ to sequences by $\mathcal{F}$.

a.) Show that $\mathcal{F}$ is a continuous mapping between $(\mathcal{C}([0,1]), \|\cdot\|_{L^1})$ and ℓ^∞.

proof:

We first prove that $\mathcal{F}$ is linear:

$$(\mathcal{F}(\lambda f + \nu g))_n = \int_0^1 (\lambda f(x) + \nu g(x))\sin(2\pi n x)\,dx$$
$$= \lambda \int_0^1 f(x)\sin(2\pi n x)\,dx + \nu \int_0^1 g(x)\sin(2\pi n x)\,dx$$
$$= \lambda(\mathcal{F}(f))_n + \nu(\mathcal{F}(g))_n.$$

Therefore, to prove continuity we need to show boundedness.

$$|(\mathcal{F}(f))_n| = \left|\int_0^1 f(x)\sin(2\pi n x)\,dx\right|$$
$$\leq \|f\|_{L^1}$$
$$\Longrightarrow \sup_n |(\mathcal{F}(f))_n| = \|\mathcal{F}(f)\|_{\ell^\infty} \leq \|f\|_{L^1}.$$

Consequently, $\mathcal{F}$ is bounded with respect to these norms.

b.) Show that $\mathcal{F}$ is a continuous mapping between $(\mathcal{C}[0,1], \|\cdot\|_{L^2})$ and ℓ^2.

proof:

Define $f_N(x) = \sum_{n=1}^N a_n \sin(n\pi x)$. Therefore,

$$0 \leq \|f - f_N\|_{L^2}^2$$
$$= \|f\|_{L^2}^2 - 2\int_0^1 \sum_{n=1}^N a_n \sin(n\pi x) f(x)\,dx + \int_0^1 \sum_{n=1}^N \sum_{m=1}^N a_n \cdot a_m \sin(n\pi x)\sin(m\pi x)\,dx$$
$$= \|f\|_{L^2}^2 - 2\int_0^1 \sum_{n=1}^N a_n \sin(n\pi x) f(x)\,dx + \int_0^1 \sum_{n=1}^N a_n^2 \sin^2(n\pi x)\,dx$$
$$= \|f\|_{L^2}^2 - 2\sum_{n=1}^N a_n \int_0^1 \sin(n\pi x) f(x)\,dx + \sum_{n=1}^N a_n^2 \int_0^1 \sin^2(n\pi x)\,dx$$
$$= \|f\|_{L^2}^2 - 2\sum_{n=1}^N a_n^2 + \sum_{n=1}^N \frac{a_n^2}{2}$$
$$\Longrightarrow \frac{3}{2}\sum_{n=1}^N a_n^2 \leq \|f\|_{L^2}^2$$

Therefore, for all $N \in \mathbb{N}$:
$$\sum_{n=1}^{N} a_n^2 \leq \frac{2}{3}\|f\|_{L^2}^2$$
Consequently,
$$\|\mathcal{F}f\|_{\ell^2} \leq \sqrt{\frac{2}{3}}\|f\|_{L^2}$$
and thus $\mathcal{F}$ is bounded with respect to these norms. ■

#3.

ℓ_c is the space of all real valued sequences that have only a finite number of non-zero terms. Recall that c_0 is the space of all sequences $a_n \in \mathbb{R}$ satisfying $\lim_{n\to\infty} a_n = 0$.

a.) If c_0 is equipped with the $\|\cdot\|_\infty$ norm, what is the dual space of c_0? That is, what is the space of bounded linear functionals?

solution:

Let $L \in B(c_0, \mathbb{R})$. Therefore, for $(a_1, a_2, \dots) \in c_0$ there exists $(\beta_1, \beta_2, \dots)$ such that
$$\begin{aligned} L(a_1, a_2, \dots) &= a_1 L(1, 0, \dots) + a_2 L(0, 1, \dots) + \dots \\ &= a_1\beta_1 + a_2\beta_2 + \dots \\ &= \sum_{i=1}^{\infty} a_i\beta_i \end{aligned}$$
$$\begin{aligned} \Rightarrow |L(a_1, a_2, \dots)| &\leq \sum_{i=1}^{\infty} |a_i| \cdot |\beta_i| \\ &\leq \|a\|_\infty \cdot \|\beta\|_1 \end{aligned}$$
Consequently, if $\beta \in \ell^1$ then L is bounded.

Now, if $a = (\text{sgn}(\beta_1)\sup(|\beta|), \text{sgn}(\beta_2)\sup(|\beta|), \dots, \text{sgn}(\beta_n)\sup(|\beta|), 0, 0, \dots)$ then
$$\frac{\left|\sum_{i=1}^{\infty} \beta_i a_i\right|}{\|a\|_\infty} = \frac{\sum_{i=1}^{N} |\beta_i| \sup(|\beta|)}{|\sup(\beta)|} = \sum_{i=1}^{N} |\beta_i| \leq \|\beta\|_{\ell^1}$$
$$\Rightarrow \sup_a \frac{\left|\sum_{i=1}^{\infty} \beta_i a_i\right|}{\|a\|_\infty} = \|\beta\|_{\ell^1}.$$
Therefore, the dual space of c_0 is ℓ^1.

c.) Show that l_c is dense in the sequence space ℓ^p with respect to the ℓ^p norm.

proof:

Let $a \in \ell^p$. Then, for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\left(\sum_{i=N}^{\infty} |a_i|^p\right)^{1/p} < \varepsilon.$$

Let $\beta \in l_c$ be defined by

$$\beta = (a_1, a_2, \ldots, a_{N-1}; 0, 0, \ldots).$$

Therefore,

$$\left(\sum_{i=1}^{\infty} |\beta_i - a_i|^p\right)^{1/p} = \left(\sum_{i=N}^{\infty} |a_i|^p\right)^{1/p} < \varepsilon.$$

d.) Show that the closure of l_c in the sup-norm is c_0.

solution:

Let $a = (a_1, a_2, \ldots) \in c_0$. Define $a^{(n)} \in l_c$ by $a^{(n)} = (a_1, a_2, \ldots, a_n, 0, 0, \ldots)$. Then, since $a \in c_0$ it follows that for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $|a_n| < \varepsilon$. Therefore, if $n \geq N$ then

$$\|a - a^{(n)}\|_\infty = \sup_{n > N} |a_n| < \varepsilon.$$

Consequently, $a^{(n)} \to a$ and thus $c_0 \subset \overline{l_c}$.

Now, if $a \in \overline{l_c}$ then there exists $a^{(n)} \in l_c$ such that $a^{(n)} \to a$. Therefore, for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N$ implies

$$\|a^{(n)} - a\|_\infty < \varepsilon.$$

Consequently, for each $n \geq N$ there exists $m_n \geq n$ such that

$$\|a^{(n)} - a\|_\infty = \sup_{m \geq m_n} |a_m| < \varepsilon.$$

Therefore, $\lim_{m \to \infty} a_m = 0$ which proves that $a \in c_0$. ∎

#4.

Show that the mapping defined by

$$f(x) \to Tf(x) = \int_0^\pi \sin(x-y) f(y)\,dy,$$

maps functions in $\mathcal{C}([0,\pi])$ into $\mathcal{C}([0,\pi])$. Is this mapping continuous with respect to the L^1 and L^∞ norms on $\mathcal{C}([0,\pi])$, and if so, what are the corresponding induced norms for T?

Solution:

1. Let $f \in \mathcal{C}([0,\pi])$. Then, $\int_0^\pi |f(y)|\,dy = M < \infty$. Now, let

$$g(x) = \int_0^\pi \sin(x-y) f(y)\,dy.$$

Let $x_0 \in [0,\pi]$ and suppose $x_n \to x_0$. Therefore,

$$\begin{aligned}|g(x_0) - g(x_n)| &= \left|\int_0^\pi (\sin(x_0-y) - \sin(x_n - y)) f(y)\,dy\right| \\ &= \left|\int_0^\pi f(y) 2\cos\left(\frac{x_0+x_n-2y}{2}\right)\sin\left(\frac{x_0-x_n}{2}\right)dy\right| \\ &\leq \left|\sin\left(\frac{x_0-x_n}{2}\right)\right| \int_0^\pi |2f(y)|\,dy \\ &\leq 2M\sin\left(\frac{x_0-x_n}{2}\right).\end{aligned}$$

Therefore, $\lim_{n\to\infty} g(x_n) = g(x_0)$, which proves g is continuous.

2. $$\begin{aligned}\|Tf\|_1 &= \int_0^\pi \left|\int_0^\pi f(y)\sin(x-y)\,dy\right| dx \\ &\leq \int_0^\pi\int_0^\pi |f(y)\sin(x-y)|\,dy\,dx \\ &= \int_0^\pi |f(y)| \int_0^\pi |\sin(x-y)|\,dx\,dy.\end{aligned}$$

Now since we are integrating over one period of $\sin(x-y)$ it follows that for all y:

$$\int_0^\pi |\sin(x-y)|\,dx = 2\int_0^{\pi/2} \sin(x)\,dx = 2.$$

Therefore,

$$\|Tf\|_1 \leq 2\|f\|_1$$

which proves T is continuous in the L^1-norm.

Also,

$$\|Tf\|_\infty = \sup_x \left|\int_0^\pi f(y)\sin(x-y)\,dy\right|$$
$$\le \sup_x \|f\|_\infty \cdot \int_0^\pi |\sin(x-y)|\,dy$$
$$= \sup_x \|f\|_\infty \cdot 2$$
$$= 2\|f\|_\infty.$$

Therefore, T is bounded and hence continuous with respect to the $\|\cdot\|_\infty$ norm.

■

#5.

Let $A=\{u\in C^1([a,b]): u(a)=\alpha \text{ and } u(b)=\beta\}$. Consider the functional $I: A\to\mathbb{R}$ defined by

$$I[u]=\int_a^b g(x,u)\sqrt{1+\left(\frac{du}{dx}\right)^2}\,dx,$$

where g is a smooth function.

a.) Calculate the weak-form of the Euler-Lagrange equations.

solution:

Let $\eta\in C_c^\infty([a,b])$. Then,

$$I[u+h\eta]=\int_a^b g(x,u+h\eta)\sqrt{1+(u'+h\eta')^2}\,dx$$

$$\Rightarrow \frac{d}{dh}I[u+h\eta]=\int_a^b \Big[g_u(x,u+h\eta)\eta\sqrt{1+(u'+h\eta')^2}+g(x,u+h\eta)\cdot\frac{1}{2}(1+(u'+h\eta')^2)^{-1/2}\cdot 2\eta'\Big]dx$$

$$\Rightarrow \frac{d}{dh}I[u+h\eta]\Big|_{h=0}=\int_a^b\left[g_u(x,u)\sqrt{1+u'^2}+\frac{g(x,u)}{\sqrt{1+u'^2}}\eta'\right]dx$$

The weak-form of the Euler-Lagrange equations is then.

$$\int_a^b\left[g_u(x,u)\sqrt{1+u'^2}+\frac{g(x,u)\eta'}{\sqrt{1+u'^2}}\right]dx=0$$

b.) Integrating by parts the strong-form of the Euler-Lagrange equations is then

$$g_u\sqrt{1+u'^2}=\frac{d}{dx}\left(\frac{g(x,u)}{\sqrt{1+u'^2}}\right)$$

■

#6.

Let $\mathcal{A}=\{v\in C^1([-1,1]): v(-1)=0 \text{ and } v(1)=1\}$. Consider the functional $I:\mathcal{A}\to\mathbb{R}$ defined by

$$I[v]=\int_{-1}^{1} v(x)^2\cdot\left(1-\left(\frac{dv}{dx}\right)\right)^2 dx.$$

Prove that I does not have a minimizer in $\mathcal{A}$.

Solution:

Let $v_n\in\mathcal{A}$ be a sequence converging to v^* defined by:

$$v^*=\begin{cases} 0, & -1\le x\le 0 \\ x, & 0\le x\le 1\end{cases}$$

Then, $\lim_{n\to\infty} I[v_n]=0$. Since $I\ge 0$ it follows that $\inf_{f\in\mathcal{A}} I[f]=0$. However, $v^*\notin\mathcal{A}$ and $I[f]=0$ ~~if~~ if and only if $f=v^*$. Therefore, I does not have a minimum in $\mathcal{A}$.