Homework #2

#1(c)

Prove that $(\mathbb{R}^n, d_\infty)$ is a metric space with metric

$$d_\infty(\vec{x},\vec{y}) = \max_{i\in\{1,\dots,n\}} |y_i - x_i|.$$

proof:

Let $\vec{x},\vec{y},\vec{z}\in\mathbb{R}^n$. I will just prove triangle inequality:

$$\begin{aligned} d_\infty(\vec{x},\vec{y}) &= \max_{i\in\{1,\dots,n\}}\{|y_i - x_i|\} \\ &\le \max_{i\in\{1,\dots,n\}}\{|y_i - z_i| + |z_i - x_i|\} \\ &\le \max_{i\in\{1,\dots,n\}}\{|y_i - z_i|\} + \max_{j\in\{1,\dots,n\}}\{|z_j - x_j|\} \\ &= d_\infty(\vec{z},\vec{y}) + d_\infty(\vec{x},\vec{z}). \end{aligned}$$

#1(d)

Let X be the set of n-letter words in a K-character alphabet $A=\{a_1,\dots,a_K\}$, meaning that $X=\{(x_1,\dots,x_n): x_i\in A\}$. We define the distance between two words $x=(x_1,\dots,x_n)$ and $y=(y_1,\dots,y_n)$ to be the number of places in which the words have different letters. That is

$$d(x,y) = \#\{i : x_i \neq y_i\}.$$

Prove that (X,d) is a metric space.

proof:

Let $\vec{x},\vec{y},\vec{z}\in X$. I will again just prove the triangle inequality. First, if $j\in\{i : x_i\neq y_i\}$ then $x_j\neq z_j$ or $y_j\neq z_j$. Therefore,

$$\{i : x_i\neq y_i\} \subset \{i : x_i\neq z_i\}\cup\{i : z_i\neq y_i\}.$$

Consequently,

$$\begin{aligned} \#\{i : x_i\neq y_i\} &\le \#\left[\{i : x_i\neq z_i\}\cup\{i : z_i\neq y_i\}\right] \\ &\le \#\{i : x_i\neq z_i\} + \#\{i : z_i\neq y_i\} \\ &= d(x,z) + d(y,z). \end{aligned}$$

#2.

a.) Prove that $(\mathbb{R}^n, d)$ is a metric space with metric

$$d(\vec{x},\vec{y}) = \begin{cases} 1, & \vec{x}\neq\vec{y} \\ 0, & \vec{x}=\vec{y} \end{cases}.$$

proof:

Let $\vec{x},\vec{y},\vec{z}\in\mathbb{R}^n$. I will prove the triangle inequality. Suppose $\vec{x}\neq\vec{y}$ then $\vec{z}$ cannot satisfy both $\vec{x}=\vec{z}$ and $\vec{y}=\vec{z}$ and therefore

$$1 = d(\vec{x},\vec{y}) \leq d(\vec{x},\vec{z}) + d(\vec{y},\vec{z}).$$

b.) Show that in this metric space every subset of $\mathbb{R}^n$ is open.

proof:

Let $V\subset\mathbb{R}^n$ and let $x\in V$. Then, $B_{1/2}(x)=\{x\}\subset V$. Therefore, V is open.

c.) Show that in this metric space a sequence converges to $x\in\mathbb{R}^n$ if and only if there exists $N\in\mathbb{N}$ such that $n\geq N$ implies $x_n=x$.

proof:

Suppose $x_n\to x$ in this metric space. Then, there exists $N\in\mathbb{N}$ such that $n\geq N$ implies $d(x_n,x)<1/2$ which implies $d(x_n,x)=0$ and therefore $x_n=x$.

#3.

(X, d_X) and (Y, d_Y) are metric spaces.

a.) If $U\subseteq X$, show that (U, d_U) is a metric space where

$$d_U(u_1,u_2) = d_X(u_1,u_2)$$

for all $u_1,u_2\in U$.

proof:

Let $u_1,u_2,u_3\in U$. Triangle inequality:

$$d_U(u_1,u_2) = d_X(u_1,u_2) \leq d_X(u_1,u_3) + d_X(u_2,u_3) = d_U(u_1,u_3) + d_U(u_2,u_3)$$

b.) Show that the Cartesian product $\mathbb{X}\times\mathbb{Y}$ is a metric space with the metric

$$d(z_1,z_2)=d_{\mathbb{X}}(x_1,x_2)+d_{\mathbb{Y}}(y_1,y_2),$$

where $z_1=(x_1,y_1)$ and $z_2=(x_2,y_2)$.

proof:

Let $z_1,z_2,z_3\in\mathbb{X}\times\mathbb{Y}$. Triangle inequality:

$$\begin{aligned} d(z_1,z_2)&=d_{\mathbb{X}}(x_1,x_2)+d_{\mathbb{Y}}(y_1,y_2)\\ &\le d_{\mathbb{X}}(x_1,x_3)+d_{\mathbb{X}}(x_3,x_2)+d_{\mathbb{Y}}(y_1,y_3)+d_{\mathbb{Y}}(y_3,y_2)\\ &=d(z_1,z_3)+d(z_3,z_2)\end{aligned}$$

#4

b.) Prove that if $f(0)\geq 0$ and f is concave then for all $t\in[0,1]$:

$$f(tx)\geq tf(x).$$

proof:

Letting $y=0$ in the definition of concavity:

$$(1-\alpha)f(x)+\alpha f(0)\le f((1-\alpha)x+\alpha\cdot 0)$$

$$\Rightarrow (1-\alpha)f(x)\le f((1-\alpha)x).$$

Let $t=(1-\alpha)$;

$$tf(x)\le f(tx).$$

c.) Prove that if $f(0)\geq 0$ and f is concave then for all $a,b\geq 0$:

$$f(a+b)\le f(a)+f(b).$$

proof:

$$f(a)=f\left((a+b)\cdot\frac{a}{a+b}\right)\geq\frac{a}{a+b}f(a+b)$$

$$f(b)=f\left((a+b)\frac{b}{a+b}\right)\geq\frac{b}{a+b}f(a+b)$$

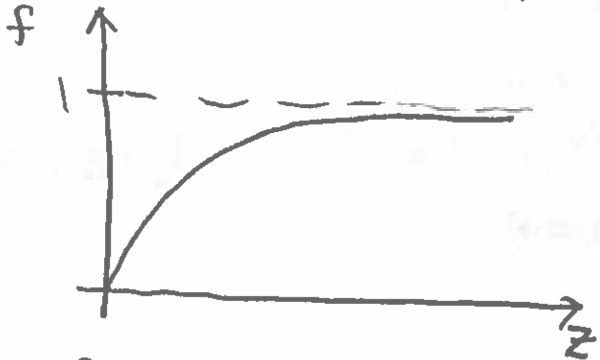
$$\Rightarrow f(a)+f(b)\geq f(a+b).$$

d.) Show that if $(\mathbb{X}, d)$ is a metric space then $(\mathbb{X}, \rho)$ is a metric space, where

$$\rho(x_1, x_2) = \frac{d(x_1, x_2)}{1 + d(x_1, x_2)}.$$

<u>proof</u>:

Let $f(z) = \frac{z}{1+z}$ which is graphed below for $z \geq 0$:



Clearly, f is an increasing, concave function. Therefore, by the triangle inequality on d:

$$d(x_1, x_2) \leq d(x_1, x_3) + d(x_2, x_3).$$

Therefore, using properties of increasing, concave functions it follows that

$$\begin{aligned}
\rho(x_1, x_2) &= f(d(x_1, x_2)) \\
&\leq f(d(x_1, x_3) + d(x_3, x_2)) \\
&\leq f(d(x_1, x_3)) + f(d(x_3, x_2)) \\
&= \rho(x_1, x_3) + \rho(x_3, x_2).
\end{aligned}$$

∎

#5.

a.) Prove that if X_n is bounded then $\liminf_{n\to\infty} X_n$ and $\limsup_{n\to\infty} X_n$ exist in the sense that they are both finite numbers.

proof:

Let $y_n = \sup\{X_k : k \geq n\}$. Then, y_n is a monotone decreasing sequence that is bounded below. Therefore, by the monotone convergence theorem:

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} \sup\{X_k : k \geq n\}$$
$$= \limsup_{n\to\infty} X_n$$

exists. A similar argument proves that $\liminf_{n\to\infty} X_n$ exists.

b.) Let C denote the set of cluster points of X_n. Prove that $\limsup_{n\to\infty} X_n = \max C$ and $\liminf_{n\to\infty} X_n = \min C$.

proof:

1. Let X_{n_k} be a subsequence such that

$$\lim_{k\to\infty} X_{n_k} = \max C$$

Therefore, for all k:

$$X_{n_k} \leq \sup\{X_\ell : \ell \geq n_k\}.$$
$$\Rightarrow \lim_{k\to\infty} X_{n_k} \leq \lim_{k\to\infty} \sup\{X_\ell : \ell \geq n_k\}$$
$$= \limsup_{k\to\infty} X_{n_k}$$
$$= \limsup_{n\to\infty} X_n$$

Therefore,

$$\max C \leq \limsup_{n\to\infty} X_n.$$

2. Let $y_n = \sup\{X_k : k \geq n\}$. Therefore, there exists a subsequence X_{n_k} such that $|X_{n_k} - y_n| < 1/k$. Consequently, $\lim_{k\to\infty} X_{n_k} = \limsup_{n\to\infty} X_n \leq \max C$.

By items 1 and 2

$$\max C = \limsup_{n\to\infty} X_n.$$