Homework 3

Analysis

Due: February 05, 2018

1. **Continuous Mappings**: A mapping between metric spaces $f : (X, d_X) \mapsto (Y, d_Y)$ is continuous if $x_n \to x$ in (X, d_X) implies that $f(x_n) \to f(x)$ in (Y, d_Y).

   (a) Propose an $\varepsilon - \delta$ definition for continuity and prove that your definition agrees with the above characterization.

   (b) Let (X, d_X), (Y, d_Y), (Z, d_Z) be metric spaces and let $f : X \mapsto Y$, and $g : Y \mapsto Z$ be continuous functions. Show that the composition

   $$h = g \circ f : X \mapsto Z,$$

   defined by $h(x) = g(f(x))$, is also continuous.

   (c) Let x_0 be a given point in a metric space (X, d). Show that the function $f : X \mapsto \mathbb{R}$ defined by $f(x) = d(x, x_0)$ is a continuous function.

2. **Completeness and Compactness**:

   (a) Let (X, d) be a metric space and suppose $K \subset X$ is compact with respect to this metric. Prove that a closed subset of K is compact.

   (b) Let (X, d) be a complete metric space, and $Y \subset X$. Prove that (Y, d) is complete if and only if Y is a closed subset of X.

   (c) Suppose that x_n is a sequence in a compact metric space with the property that every convergent subsequence has the same limit x. Prove that $x_n \to x$ as $n \to \infty$.

   (d) Let $f : (X, d_X) \mapsto (Y, d_Y)$ be a continuous mapping and assume X is compact. Prove that the range $f(X)$ is a compact subset of Y.

   (e) Let $f : (X, d) \mapsto \mathbb{R}$ be a continuous mapping and assume X is compact. Prove that f is bounded and obtains its maximum and minimum values.

3. **Geometry of Norms**:

   (a) What is the largest r for which the l^2 circle $\{\mathbf{x} \in \mathbb{R}^2 : \|x\|_2 = r\}$ fits into the the l^1 unit ball on $\mathbb{R}^2$. What is the "radius" of the largest l^1 circle $\{\mathbf{x} : \|x\|_1 = r\}$ that will fit into the l^2 unit ball on $\mathbb{R}^2$.

   (b) Prove that

   $$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = \left(|x_1 - y_1|^{\frac{1}{3}} + |x_2 - y_2|^{\frac{1}{3}}\right)^3$$

   is not a metric on $\mathbb{R}^2$.

   (c) Take the points $\mathbf{x} = (1, 0)$, $\mathbf{y} = (0, 1)$. Let d be one of the ~~metrics~~ l^p metrics for $p \geq 1$, or let it be the corresponding non-metric for $0 < p < 1$. Show that

   $$\begin{cases} d(\mathbf{x}, \mathbf{y}) < \mathbf{d}(\mathbf{x}, 0) + d(0, \mathbf{y}) & \text{if } p > 1 \\ d(\mathbf{x}, \mathbf{y}) = \mathbf{d}(\mathbf{x}, 0) + d(0, \mathbf{y}) & \text{if } p = 1 \\ d(\mathbf{x}, \mathbf{y}) > \mathbf{d}(\mathbf{x}, 0) + d(0, \mathbf{y}) & \text{if } p < 1 \end{cases}.$$

   Draw the unit balls centered on the origin, on $\mathbf{x}$, and on $\mathbf{y}$ in each of the three cases. Notice how they "bend" the wrong way when $p < 1$.

(d) Find the shortest distance, in the l^1 metric on $\mathbb{R}^2$, from the origin to the line $x_1 + x_2 = 2$. In the l^∞ metric. In the l^2 metric. Is the shortest distance in the l^p metric a monotone function of p?

(e) Let $(X, \|\cdot\|)$ be a normed linear space. A set $C \subset X$ is convex if for all $x, y \in C$ and all real numbers $0 \leq t \leq 1$:

$$tx + (1-t)y \in C,$$

i.e. the line segment joining any two points in the C lies in C. Prove that unit ball with respect to $\|\cdot\|$ is convex.

4. **Equivalence of Norms:** Let $\mathbf{x} \in \mathbb{R}^n$.

(a) Show that

$$\lim_{p\to\infty} \|\mathbf{x}\|_p = \max\{|x_1|, \dots \|x_n\|\}.$$

(b) Show that, for all $p, q \geq 1$

$$\frac{\|\mathbf{x}\|_p}{n} \leq \|\mathbf{x}\|_q \leq n\|\mathbf{x}\|_p.$$

(c) Show that for all $p, q \geq 1$, if $\mathbf{x}_n \to \mathbf{x}$ with respect to the norm $\|\cdot\|_p$ then $\mathbf{x}_n \to \mathbf{x}$ with respect to the norm $\|\cdot\|_q$.

Homework #3

#1. b.

Let (X, d_X), (Y, d_Y), (Z, d_Z) be metric spaces and let $f: X \to Y$, and $g: Y \mapsto Z$ be continuous functions. Show that the composition

$$h = g \circ f : X \mapsto Z$$

defined by $h(x) = g(f(x))$, is also continuous.

~~proof~~:

Let $x_n \in X$ satisfy $x_n \xrightarrow{d_X} x$. Therefore, by continuity of f the sequence $y_n \in Y$ defined by $y_n = f(x_n)$ satisfies $y_n \xrightarrow{d_Y} y = f(x)$. Moreover, by continuity of g the sequence $z_n \in Z$ defined by $z_n = g(y_n)$ satisfies $z_n \xrightarrow{d_Z} z = g(y) = g(f(x))$. Therefore,

$$\begin{aligned}
\lim_{n\to\infty} h(x_n) &= \lim_{n\to\infty} g(f(x_n)) \\
&= \lim_{n\to\infty} g(y_n) \\
&= g(y) \\
&= g(f(x)) \\
&= h(x).
\end{aligned}$$

#1. c.

Let x_0 be a given point in a metric space (X, d). Show that $f: X \mapsto \mathbb{R}$ defined by $f(x) = d(x, x_0)$ is a continuous function.

proof:

Let $x_n \xrightarrow{d} x$. Therefore, $\lim_{n\to\infty} d(x_n, x) = 0$. By the reverse triangle

$$d(x_n, x) \geq |d(x_n, x_0) - d(x, x_0)|.$$

Therefore,

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} d(x_n, x_0) = d(x, x_0) = f(x).$$

■

#2.

a.) Let (X, d) be a metric space and suppose $K \subset X$ is compact with respect to this metric. Prove that a closed subset of K is compact.

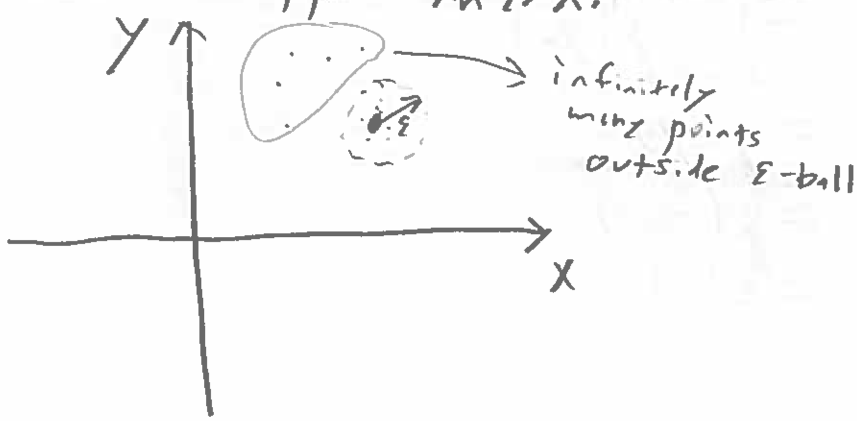
proof:

Let $V \subset K$ be closed. Let $x_n \in V$. Since K is compact it follows that x_n has a convergent subsequence. Since V is closed it follows that the limit point of x_{n_k} is in V.

C.) Suppose that x_n is a sequence in a compact metric space with the property that every convergent subsequence has the same limit x. Prove that $x_n \to x$ as $n \to \infty$.

proof:

For contradiction suppose $x_n \not\to x$.



Therefore, there exists $\varepsilon > 0$ such that for all $N \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $n \geq N$ implies $d(x_n, x) \geq \varepsilon$. (negation of convergence)

Define a subsequence by x_{n_k} is the sequence of elements satisfying $d(x_{n_k}, x) \geq \varepsilon$. Clearly $x_{n_k} \not\to x$. By compactness x_{n_k} has a further subsequence $x_{n_{k_\ell}}$ which converges to x, which is a contradiction.

d.) Let $f:(X,d_X)\mapsto(Y,d_Y)$ be a continuous mapping and assume X is compact. Prove that the range $f(X)$ is a compact subset of Y.

proof:

Let $y_n \in f(X)$. Therefore, for all $n\in\mathbb{N}$ there exists $n\in\mathbb{N}$ such that $f(x_n)=y_n$. Since X is compact there exists $x\in X$ and a subsequence x_{n_k} such that $x_{n_k}\to x$. Therefore, from continuity it follows that the subsequence $y_{n_k}=f(x_{n_k})$ satisfies

$$\lim_{k\to\infty} y_{n_k}=\lim_{k\to\infty} f(x_{n_k})=f(x)\in f(X).$$

e.) Let $f:(X,d)\mapsto\mathbb{R}$ be a continuous mapping and assume X is compact. Prove that f is bounded and obtains it maximum and minimum values.

proof:

By the previous problem $f(X)$ is compact and hence closed and bounded. Therefore, $\sup\{y\in f(X)\}<\infty$ and there exists a sequence $y_n\to\sup\{y\in f(X)\}$. Since $f(X)$ is closed it follows that $\sup\{y\in f(X)\}\in f(X)$. A similar argument proves the minimum is obtained. ∎

#3.

b.) Prove that

$$d(\vec{x},\vec{y}) = \|x-y\| = (|x_1-y_1|^{1/3} + |x_2-y_2|^{1/3})^3$$

is not a metric on $\mathbb{R}^2$.

proof:

Let $\vec{x}=(1,0)$ and $\vec{y}=(0,1)$. Then,

$$d(\vec{x},\vec{y}) = 8.$$

However,

$$d(\vec{x},0) = 1 \text{ and } d(0,\vec{y}) = 1.$$

Therefore,

$$d(\vec{x},\vec{y}) > d(\vec{x},0) + d(0,\vec{y}).$$

c.) Take the points $\vec{x}=(1,0)$, $\vec{y}=(0,1)$. Let d be one of the ℓ^p metrics for $p \geq 1$, or let it be the corresponding non-metric for $0<p<1$. Show that

$$\begin{cases} d(x,y) < d(x,0)+d(0,y) & \text{if } p>1 \\ d(x,y) = d(x,0)+d(0,y) & \text{if } p=1 \\ d(x,y) > d(x,0)+d(0,y) & \text{if } p<1. \end{cases}$$

Draw the unit balls centered on the origin, on x, and on y in each of the cases.

proof:

Calculating,

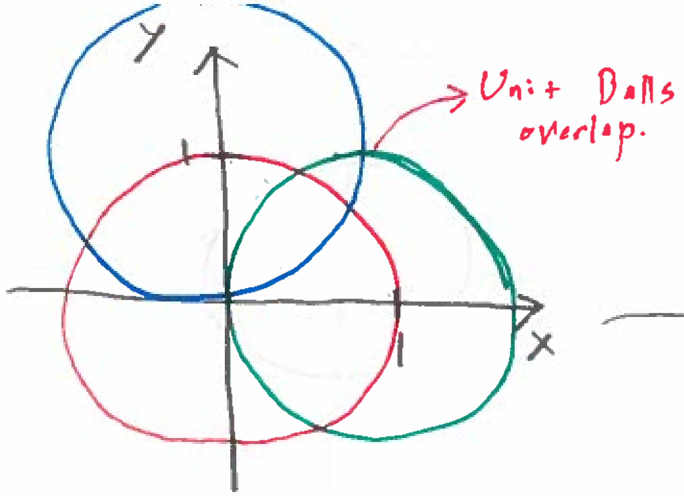
$$d(x,y) = 2^{1/p}$$

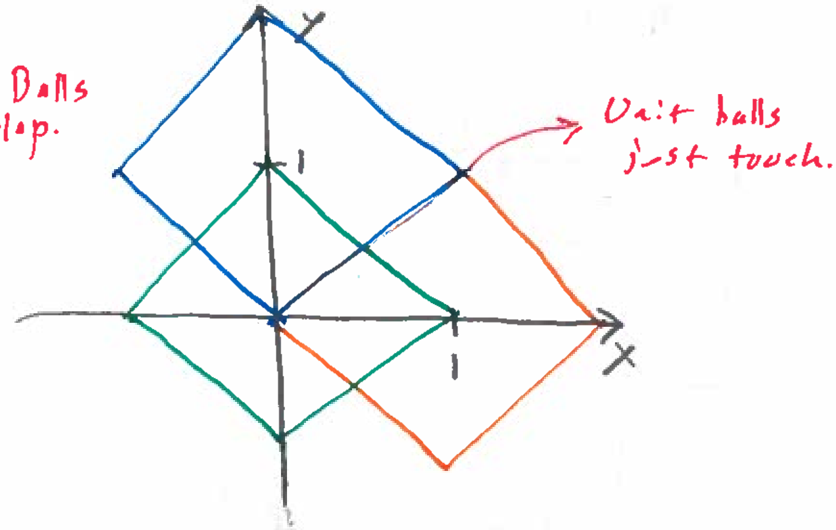
while

$$d(x,0) + d(0,y) = 2.$$

Now

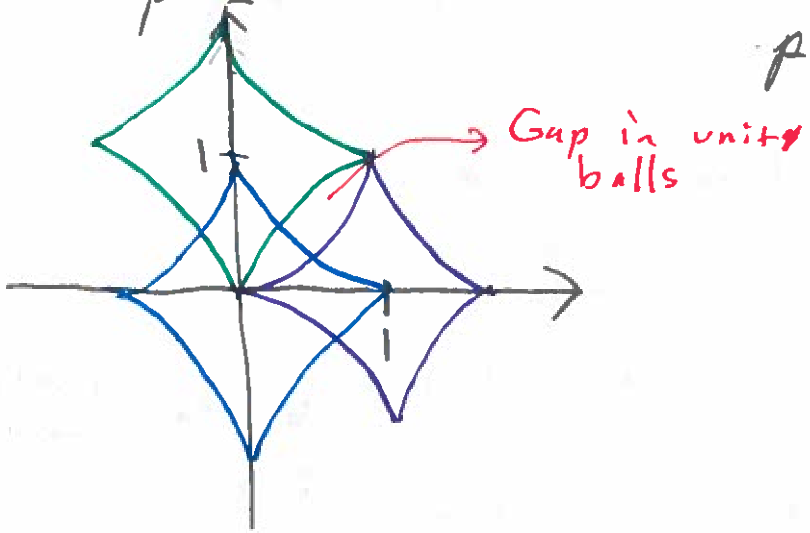
$$\begin{cases} 2^{1/p} < 2 & \text{if } p>1 \\ 2^{1/p} = 2 & \text{if } p=1 \\ 2^{1/p} > 2 & \text{if } p<1. \end{cases}$$





$p=2$

$p=1$

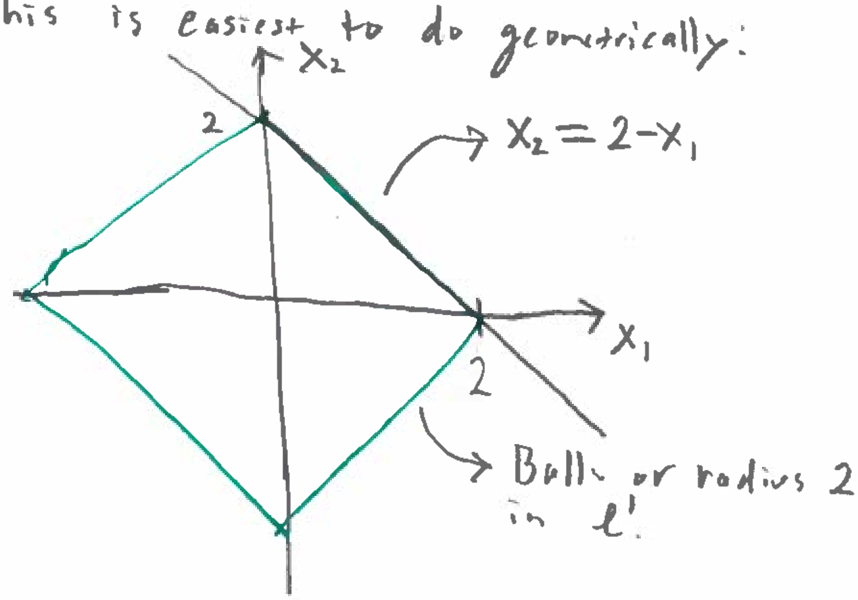


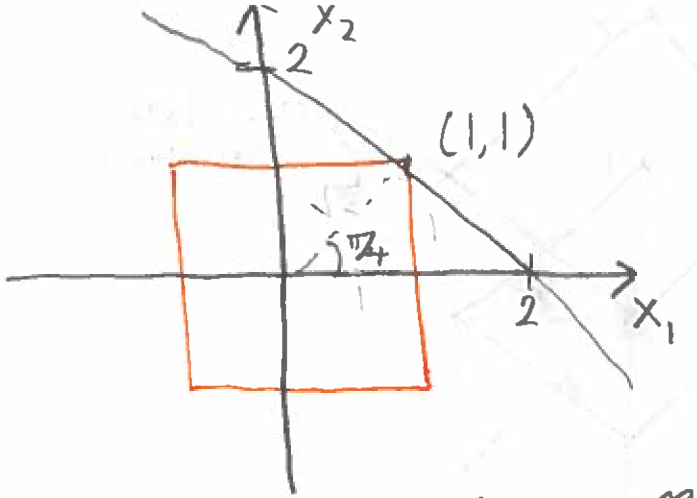
$p=1/2$

d.) Find the shortest distance in the ℓ^1 metric from the origin to the line $x_1+x_2=2$. In the ℓ^∞ metric. In the ℓ^2 metric. Is the shortest distance in the ℓ^p metric a monotone function of p?

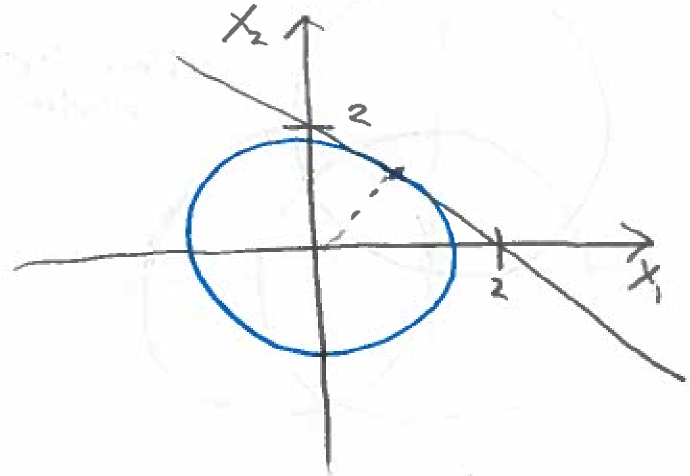
<u>solution</u>:

This is easiest to do geometrically:





Ball of radius 1 in ℓ^∞.



Ball of radius $\sqrt{2}$ in ℓ^2.

e.) Let $(X, \|\cdot\|)$ be a normed linear space. A set $C \subset X$ is convex if for all $x, y \in C$ and all $0 \le t \le 1$:

$$tx + (1-t)y \in C$$

Prove that the unit ball with respect to $\|\cdot\|$ is convex

proof:

Let $x, y \in B_1(0)$. Let $z = tx + (1-t)y$. Then,

$$\begin{aligned} \|z\| = \|tx + (1-t)y\| &\le \|tx\| + \|(1-t)y\| \\ &= t\|x\| + (1-t)\|y\| \\ &\le t + (1-t) \\ &= 1. \end{aligned}$$

$\blacksquare$

#4.

Let $x \in \mathbb{R}^n$.

a.) Show that

$$\lim_{p\to\infty} \|x\|_p = \|x\|_\infty.$$

proof:

Since $\|x\|_p$ is monotone in p it follows that $\lim_{p\to\infty}\|x\|_p$ exists. Let $L = \lim_{p\to\infty}\|x\|_p$. Therefore, by continuity of $\ln$ it follows that

$$\begin{aligned}
\ln(L) &= \lim_{p\to\infty} \frac{1}{p}\ln\left(\sum_{i=1}^{n} |x_i|^p\right)\\
&= \lim_{p\to\infty} \frac{1}{p}\ln\left(\|x\|_\infty^p \sum_{i=1}^{n} \left|\frac{x_i}{\|x\|_\infty}\right|^p\right)\\
&= \lim_{p\to\infty} \left[\ln(\|x\|_\infty) + \frac{1}{p}\ln\left(\sum_{i=1}^{n}\left|\frac{x_i}{\|x\|_\infty}\right|^p\right)\right]\\
&= \ln(\|x\|_\infty) + 0
\end{aligned}$$

$$\Rightarrow L = \|x\|_\infty$$

$\blacksquare$

b.) Show that, for all $p, q \geq 1$:

$$\frac{\|x\|_p}{n} \leq \|x\|_q \leq n\|x\|_p$$

proof:

For all p:

1. $\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \leq \left(\sum_{i=1}^{n} \|x\|_\infty^p\right)^{1/p} = n^{1/p}\|x\|_\infty \leq n\|x\|_\infty.$
2. $\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \geq \left(\sum_{i=1}^{n} \|x\|_\infty^p\right)^{1/p} = \|x\|_\infty.$

By items 1 and 2 it follows that

$$\|x\|_q \leq n\|x\|_\infty \leq n\|x\|_p \quad \text{and} \quad \|x\|_q \geq \|x\|_\infty \geq \frac{1}{n}\|x\|_p$$

$$\Rightarrow \frac{1}{n}\|x\|_p \leq \|x\|_q \leq n\|x\|_p.$$

$\blacksquare$

C.) Show that for all $p, q \geq 1$, if $x_n \to x$ with respect to $\|\cdot\|_p$ then $x_n \to x$ with respect to $\|\cdot\|_q$.

proof:

This follows from part b.