Homework 4

Analysis

Due: February 12, 2018

1. **Linearity:**

   (a) Let V be the set of continuous real-valued functions on $[0,1]$ satisfying $\int_0^1 xf(x)\,dx = 0$. Is V a linear space?

   (b) Let V be the set of real-valued, twice differentiable functions on $[0,1]$ that are solutions of the differential equation
   $$y''(x) + xy(x) = 0.$$
   Is V a linear space?

   (c) Let V be the set of real-valued, twice differentiable functions on $[0,1]$ that are solutions of the differential equation
   $$y''(x) + xy(x) = \sin(x).$$
   Is V a linear space?

2. **Completeness and Compactness:**

   (a) Show that $\mathbb{R}^n$ is a complete metric space in the l^p metric for $p \geq 1$ (including $p = \infty$).

   (b) Show that $l^1(\mathbb{N}, \mathbb{R})$ is a complete metric space.

   (c) Show that the set $\{x \in l^1(\mathbb{N}, \mathbb{R}) : \|x\|_1 \leq 1\}$, is not compact with respect to the l^1 norm.

3. **Minimization:**

   (a) A function $f : \mathbb{R}^n \mapsto \mathbb{R}$ is coercive if
   $$\lim_{\|x\| \to \infty} f(x) = \infty.$$
   Explicitly, this condition means that for any $M > 0$ there is an $R > 0$ such that $\|x\| \geq R$ implies $f(x) \geq M$. Prove that if f is lower semicontinuous with respect to this metric and coercive, then f is bounded from below and attains its infimum.

   (b) Let $p : \mathbb{R}^2 \mapsto \mathbb{R}$ be a polynomial function of two real variables. Suppose that $p(x, y) \geq 0$ for all $x, y \in \mathbb{R}$. Does every such function attain its infimum? Prove or disprove.

4. **Matrix Norms:**

   (a) Let $A \in \mathbb{R}^{n \times n}$. Prove that the following definitions of a matrix norm are identical:

   - $\|A\| = \max\limits_{\mathbf{x} \in \mathbf{R}^n} \dfrac{\|A\mathbf{x}\|}{\|\mathbf{x}\|}$.
   - $\|A\| = \max\limits_{\|\mathbf{x}\| = 1} \|A\mathbf{x}\|$.
   - $\|A\| = \max\limits_{\|\mathbf{x}\| \leq 1} \dfrac{\|A\mathbf{x}\|}{\|\mathbf{x}\|}$.

Note: The norm on $\mathbb{R}^n$ does not matter.

(b) Suppose $D \in \mathbb{R}^{n\times n}$ is a diagonal matrix with diagonal entries $d_1, \ldots, d_n$. Prove that for all $p \in [1, \infty]$:

$$\|D\|_p = \max_{1\leq i\leq n} |d_i|.$$

(c) Show that for all $A \in \mathbb{R}^{n\times n}$ that

$$\|A\|_\infty \leq \sqrt{n}\|A\|_2.$$

(d) For a matrix $A \in \mathbb{R}^{n\times n}$ with entries a_{ij} show that

- $\|A\|_\infty \leq \max_i \sum_{j=1}^{n} |a_{ij}|$.
- $\|A\|_1 \leq \max_j \sum_{i=1}^{n} |a_{ij}|$.
- Show that there always exists $x \in \mathbb{R}^n$ for which $\|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|$. Deduce that $\|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|$.
- Prove in a similar way $\|A\|_1 = \max_j \sum_{i=1}^{n} |a_{ij}|$.

5. **Sequence spaces:** $c_0 \subset \mathbb{R}^{\mathbb{N}}$ is the set of real valued sequences $\{x_n\}$ such that $\lim_{n\to\infty} x_n = 0$.

(a) If $x, y \in c_0$, show that $x + y \in c_0$.

(b) Show that $l^p(\mathbb{N}, \mathbb{R}) \subset c_0$ for all $1 \leq p < \infty$ and that the containment is strict.

(c) Show the c_0 is complete in the l^∞ norm.

Homework #4

#1.

b.) Let V be the set of real-valued twice differentiable functions on $[0,1]$ that solve

$$y''(x)+xy(x)=0.$$

Show that V is a linear space.

Solution:

Yes, let $h=f+\lambda g$, where $\lambda\in\mathbb{R}$ and f,g both satisfy the differential equation.

$$\Rightarrow h''+xh=(f''+\lambda g'')+x(f+\lambda g)$$
$$=f''+xf+\lambda(g''+xg)$$
$$=0.$$

Other properties of linearity are trivial.

C.) Let V be the set of real-valued twice differentiable functions on $[0,1]$ that are solutions of the differential equation

$$y''+xy=\sin(x).$$

Is V a linear space?

Solution:

No. $0\notin V$.

#2.

a.) Show that $\mathbb{R}^n$ is a complete metric space in the ℓ^p metric for $p\geq 1$ (including $p=\infty$).

~~proof~~:

Let $x^{(m)}\in\ell^p$ be a Cauchy sequence. Then the components $x_i^{(m)}$ are Cauchy in $\mathbb{R}$ and satisfy $x_j^{(m)}\to x_j$. Let $x^*=(x_1,\ldots,x_n)$. Then, by continuity it follows that

$$d(x^{(m)},x^*)=\left(\sum_{i=1}^{n}|x_i^{(m)}-x_i|^p\right)^{1/p}\to 0.$$

A similar result holds for the ℓ^∞ norm.

b.) Show that $\ell^1(\mathbb{N},\mathbb{R})$ is a complete metric space.

<u>proof:</u>

Let $X^{(n)} \in \ell^1(\mathbb{N},\mathbb{R})$ be a Cauchy sequence. Therefore, its components $X_i^{(n)}$ are Cauchy and satisfy

$$X_i^{(n)} \to X_i.$$

Let $X^* = (X_1, X_2, \ldots)$. Therefore, for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $m, n > N$ implies

$$\sum_{i=0}^{M} |X_i^{(n)} - X_i^{(m)}| \leq \sum_{i=0}^{\infty} |X_i^{(n)} - X_i^{(m)}| < \varepsilon$$

$$\Rightarrow \lim_{m\to\infty} \sum_{i=0}^{M} |X_i^{(n)} - X_i^{(m)}| = \sum_{i=0}^{M} |X_i^{(n)} - X_i| < \varepsilon.$$

$$\Rightarrow \lim_{M\to\infty} \sum_{i=0}^{M} |X_i^{(n)} - X_i| = \sum_{i=1}^{\infty} |X_i^{(n)} - X_i| < \varepsilon$$

Finally, by Minkowski's inequality:

$$\|X\|_1 = \|X + X^{(n)} - X^{(n)}\|_1 \leq \|X^{(n)}\|_1 + \|X - X^{(n)}\|_1.$$

Since $X^{(n)}$ is Cauchy it follows that $\|X^{(n)}\|_1$ is bounded. Therefore, $X \in \ell^1(\mathbb{R},\mathbb{N})$.

C.) Show that the set $\{X \in \ell^1(\mathbb{N},\mathbb{R}) : \|X\|_1 \leq 1\}$ is not compact with respect to the ℓ^1 norm.

<u>Solution:</u>

Consider the sequence $X^{(n)} \in \ell^1(\mathbb{N},\mathbb{R})$ by

$$X_j^{(i)} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

Therefore, for all $k > i$ it follows that

$$\|X^{(k)} - X^{(i)}\| = 2,$$

hence no subsequences of $X^{(n)}$ can be Cauchy.

■

#3.

a.) A function $f:\mathbb{R}^n \to \mathbb{R}$ is coercive if

$$\lim_{\|x\|\to\infty} f(x) = \infty.$$

Prove that if f is lower semicontinuous with respect to this metric and coercive then f is bounded from below and attains its infimum.

proof:

First, since f is coercive it follows that if $\inf_{x\in\mathbb{R}^n} f(x) = -\infty$ then there exists x_0 and a sequence x_n such that $x_n \to x_0$ and $f(x_n) \to -\infty$. However, from l.s.c. it follows that

$$f(x_0) \le \liminf_{x_n \to x_0} f(x_n) = -\infty$$

which is a contradiction. Consequently, f is bounded below.

Now, let x_n be a minimizing sequence of f. That is, x_n satisfies

$$\lim_{n\to\infty} f(x_n) = \inf_{x\in\mathbb{R}^n} f(x).$$

Since, f is coercive it follows that $\|x_n\|$ is bounded, i.e. there exists $M>0$ such that $\|x_n\| \le M$. Consequently, since $K = \{x \in \mathbb{R}^n : \|x\| \le M\}$ is compact it follows that there exists $x^* \in \mathbb{R}^n$ and a subsequence $x_{n_k} \to x^*$. Therefore, by l.s.c.

$$f(x^*) \le \liminf_{k\to\infty} f(x_{n_k}) = \lim_{n\to\infty} f(x_n) = \inf_{x\in\mathbb{R}^n} f(x)$$

$$\Rightarrow f(x^*) = \inf_{x\in\mathbb{R}^n} f(x).$$

b.) Let $p:\mathbb{R}^2 \to \mathbb{R}$ be a polynomial of two real variables. Suppose $p(x,y) \ge 0$ for all $x, y \in \mathbb{R}^n$. Does every such function obtains its infimum.

proof:

Yes, since $p \ge 0$ it follows from properties of polynomials that f is coercive. Since f is continuous it follows that f is l.s.c. Therefore, by part a, p obtains its minimum.

■

#4

b.) Suppose $D \in \mathbb{R}^{n\times n}$ is a diagonal matrix with diagonal entries $d_1, \dots, d_n$. Prove that for all $p \in [1, \infty]$

$$\|D\|_p = \max_{1\le i\le n} |d_i|.$$

proof:

$$\|D\|_p = \max_{\|x\|_p=1} \|Dx\|_p = \max_{\|x\|_p=1} \Big(\sum_{i=1}^{n} |d_i x_i|^p\Big)^{1/p}$$

$$\le \max_{\|x\|_p=1} \Big(\sum_{i=1}^{n} \max_i(|d_i|^p)|x_i|^p\Big)^{1/p}$$

$$= \max_{\|x\|_p=1} \max_i |d_i| \Big(\sum_{i=1}^{n} |x_i|^p\Big)^{1/p}$$

$$= \max_i |d_i|.$$

To show equality define

$$x_i^* = \begin{cases} 0 & \text{if } i \ne j, \\ 1 & \text{if } i = j, \end{cases}$$

where j denotes the index in which $\max_i |d_i|$ is obtained. Therefore,

$$\|Dx^*\|_p = |d_j| = \max_i |d_i|.$$

c.) Show that for all $A \in \mathbb{R}^{n\times n}$ that

$$\|A\|_\infty \le \sqrt{n}\,\|A\|_2$$

proof:

First, note that for $y \in \mathbb{R}^n$

$$\|y\|_\infty \le \|y\|_2 \le \sqrt{n}\,\|y\|_\infty. \quad (*)$$

Let $y \in \mathbb{R}^n$ satisfy

$$\max_{x\in\mathbb{R}^n} \frac{\|Ax\|_\infty}{\|x\|_\infty} = \frac{\|Ay\|_\infty}{\|y\|_\infty}.$$

Consequently, by $(*)$ it follows that

$$\|A\|_\infty = \frac{\|Ay\|_\infty}{\|y\|_\infty} \le \frac{\|Ay\|_2}{\|y\|_2/\sqrt{n}} = \sqrt{n}\,\frac{\|Ay\|_2}{\|y\|_2} \le \max_{x\in\mathbb{R}^n} \sqrt{n}\,\frac{\|Ax\|_2}{\|x\|_2} = \sqrt{n}\,\|A\|_2.$$

d.) For a matrix $A \in \mathbb{R}^{n \times n}$ with entries a_{ij} show that

- $\|A\|_\infty = \max_i \sum_{j=1}^n |a_{ij}|$
- $\|A\|_1 = \max_j \sum_{i=1}^n |a_{ij}|$

proof:

1. $\|A\|_\infty = \max_{\|x\|_\infty = 1} \|Ax\|_\infty = \max_i \left| \sum_{j=1}^n a_{ij} x_j \right| \leq \max_i \sum_{j=1}^n |a_{ij}| \cdot |x_j| \leq \max_i \sum_{j=1}^n |a_{ij}|.$

Let k denote the value satisfying $\sum_{j=1}^n |a_{kj}| = \max_i \sum_{j=1}^n |a_{ij}|$ and let

$$y_j = \operatorname{sgn}(a_{kj})$$

Then,

$$\|Ay\|_\infty = \max_i \left| \sum_{j=1}^n a_{ij} y_j \right| = \max_i \sum_{j=1}^n |a_{ij}|.$$

2. $\|A\|_1 = \max_{\|x\|_1 = 1} \|Ax\|_1 = \max_{\|x\|_1 = 1} \sum_{i=1}^n \left| \sum_{j=1}^n a_{ij} x_j \right|$

$$\leq \max_{\|x\|_1 = 1} \sum_{i=1}^n \sum_{j=1}^n |a_{ij}| \cdot |x_j|$$

$$= \max_{\|x\|_1 = 1} \sum_{j=1}^n \sum_{i=1}^n |a_{ij}| \cdot |x_j|$$

$$\leq \max_j \sum_{i=1}^n |a_{ij}|.$$

To show equality let

$$y_j = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{o.w.} \end{cases}$$

where k satisfies

$$\max_j \sum_{i=1}^n |a_{ij}| = \sum_{i=1}^n |a_{ik}|.$$

It follows that

$$\|Ay\|_1 = \max_j \sum_{i=1}^n |a_{ij}|.$$

#5.

c.) Show that c_0 is complete in the ℓ^∞ norm.

proof:

Let $x^{(n)}$ be Cauchy in c_0. Then there exists $N \in \mathbb{N}$ such that $n, m \geq N$ implies

$$\|x^{(n)} - x^{(m)}\| < \varepsilon.$$

Therefore, $x_i^{(n)}$ is Cauchy and that there exists x_i such that $x_i^{(n)} \to x_i$.
Let $x^* = (x_1, x_2, \ldots)$. Now, for all $k \in \mathbb{N}$

$$\max_{1 \leq i \leq k} |x_i^{(n)} - x_i^{(m)}| \leq \|x^{(n)} - x^{(m)}\|_\infty < \varepsilon.$$

Therefore, taking $m \to \infty$ we have

$$\max_{1 \leq i \leq k} |x_i^{(n)} - x_i| < \varepsilon$$

Consequently,

$$\|x^{(n)} - x^*\|_\infty < \varepsilon.$$

and thus $x^{(n)} \to x^*$ in the ℓ^∞ norm.

Finally, we need to show $\lim_{i \to \infty} x_i = 0$. It follows that

$$|x_i| = |x_i^{(n)} - x_i| + |x_i^{(n)}|.$$

Let $n \in \mathbb{N}$ satisfy $\|x^{(n)} - x^*\|_\infty < \varepsilon/2$ and let $N \in \mathbb{N}$ such that $i \geq N$ implies $|x_i^{(n)}| < \varepsilon/2$.