# Homework 5

Analysis

Due: February 23, 2018

1. **Sequence Spaces:**
   (a) Given $1 \leq p < q < \infty$ give an example of a sequence $x_n$ which is in $l^q$ but not in $l^p$.
   (b) Given $1 \leq p < q < \infty$ prove that $l^p \subset l^\infty$.
   (c) Given $1 \leq p < q < \infty$ prove that $l^p \subset l^q$.
   (d) Let $\mathbf{x}$ be a real valued sequence with components $x_n$. Suppose for $q$ satisfying $1 \leq q < \infty$ we know that for every $\mathbf{y} \in l^q$ with components $y_i$ the sequence $x_i y_i$ is absolutely summable, and
   $$\left| \sum_{i=1}^{\infty} x_i y_i \right| \leq C \|y\|_{l^q}$$
   for some constant $C$. Show that $x \in l^p$ where $p = q/(q-1)$.
2. **Young's Inequality:**
   (a) Let $f$ be a continuous strictly increasing function on $[0, \infty)$ with $f(0) = 0$. Prove that for $a, b > 0$:
   $$ab \leq \int_0^a f(t)\, dt + \int_0^b f^{-1}(t)\, dt.$$
   (b) Prove that for all $a, b \geq 0$:
   $$\exp(a) + (1+b)\ln(1+b) \geq (1+a)(1+b).$$
3. **Hölder's and Minkowski's inequalities on spaces of functions:**
   (a) If $f, g \in C([0,1])$, show that
   $$\int_0^1 |f(t)g(t)|\, dt \leq \|f\|_{L^p} \|g\|_{L^q}$$
   with $1/p + 1/q = 1$, $1 < p < \infty$, where
   $$\|h\|_{L^m} = \left( \int_0^1 |h(t)|^m\, dt \right)^{1/m}.$$
   (b) If $f, g \in C([0,1])$ and $1 < p < \infty$, prove that
   $$\|f+g\|_p \leq \|f\|_p + \|g\|_p.$$
4. **Function Spaces:**
   (a) Let $f_n \in C([a,b])$ be a sequence of functions converging uniformly to a function $f$. Show that
   $$\lim_{n\to\infty} \int_a^b f_n(x)\, dx = \int_a^b f(x)\, dx.$$

(b) Consider the space of continuously differentiable functions:

$$C^1([a,b]) = \{f : [a,b] \mapsto \mathbb{R} : f, f' \in C([a,b])\}.$$

with the $C^1$ norm:

$$\|f\|_{C^1} = \sup_{a \leq x \leq b} |f(x)| + \sup_{a \leq x \leq b} |f'(x)|.$$

Prove that $C^1([a,b])$ is a Banach space.

(c) Show that the space $C([a,b])$ with the $L^1$ norm $\|\cdot\|_{L^1}$ defined by

$$\|f\|_{L^1} = \int_a^b |f(x)|\,dx,$$

is incomplete. Show that if $f_n \to f$ with respect to $\|\cdot\|_{L^\infty}$, then $f_n \to f$ with respect to $\|\cdot\|_{L^1}$.

5. **Lebesgue Spaces:**

(a) For each of the following functions $f$, find the set of $p \in [1,\infty]$ for which $f \in L_0^p$ on the given interval.

- $f(t) = e^{-t}$ on $[0,\infty)$.
- $f(t) = 1/t$ on $[1,\infty)$.
- $f(t) = 1/\sqrt{t}$ on $(0,1]$.
- $f(t) = \ln(t)$ on $(0,3]$.
- $f(t) = 1/\sqrt{t}$ on $(0,\infty)$.
- $f(t) = \begin{cases} |t|^{-1/2} & |t| \leq 1 \\ 2t^{-2} & |t| > 1 \end{cases}$ on $(0,\infty)$.

(b) For each of the following functions $f$, find the set of $\alpha \in \mathbb{R}$ for which $f \in L_0^p$ on the given interval, taking i) $p = 1$, ii) $p = 2$, iii) $p = \infty$.

- $f(t) = t^\alpha$ on $(0,1]$.
- $f(t) = t^\alpha$ on $[1,\infty)$.
- $f(t) = t^\alpha$ on $(0,\infty)$.
- $f(t) = \begin{cases} t^\alpha & 0 < t \leq 1 \\ t^{-\alpha} & 1 < t < \infty \end{cases}$ on $[1,\infty)$.

(c) Is there a function $f$ on $(0,\infty)$ which belongs to $L_0^1$ but not to $L_0^2$. Is there one that belongs to $L_0^2$ but not to $L_0^1$?

(d) Is there a function $f$ on $(0,1)$ which belongs to $L_0^1$ but not to $L_0^2$. Is there one that belongs to $L_0^2$ but not to $L_0^1$?

# Homework #5

## #1.

a) Given $1 \le p < q < \infty$ give an example of a sequence $x_n$ in $\ell^q$ but not in $\ell^p$.

Solution:

Let $x_n = (1/n)^{1/p}$. Therefore, $|x_n|^q = \left(\frac{1}{n}\right)^{q/p}$ but $|x_n|^p = 1/n$.

b.) Given $1 \le p < q < \infty$ prove that $\ell^p \subset \ell^\infty$.

proof:

If $x_n \in \ell^p$ then $\lim_{n\to\infty} x_n = 0$ and therefore $|x_n|$ is bounded.

c.) Given $1 \le p < q < \infty$, prove that $\ell^p \subset \ell^q$.

proof:

Let $x_n \in \ell^p$. Therefore,

$$\sum_{n=1}^{\infty} |x_n|^q = \sum_{n=1}^{\infty} |x_n|^p \cdot |x_n|^{q-p} \le \sum_{n=1}^{\infty} |x_n|^p \cdot \|x_n\|_\infty^{q-p} = \|x_n\|_\infty^{q-p} \sum_{n=1}^{\infty} |x_n|^p < \infty.$$

d.) Let $x_n$ be a real valued sequence. Suppose for $1 \le q < \infty$ we know that for every $y \in \ell^q$ the sequence $x_i y_i$ is absolutely summable, and

$$\left|\sum_{i=1}^{\infty} x_i y_i\right| \le C \|y\|_{\ell^q}$$

for some constant $C$. Show that $x \in \ell^p$ where $p = q/q-1$.

proof:

Let $y^{(n)} \in \ell^q$ be defined by

$$y^{(n)} = (\mathrm{sgn}(x_1)|x_1|^{p-1}, \mathrm{sgn}(x_2)|x_2|^{p-1}, \dots, \mathrm{sgn}(x_n)|x_n|^{p-1}, 0, 0, \dots)$$

Therefore,

$$\left|\sum_{i=1}^{\infty} x_i y_i\right| = \sum_{i=1}^{n} |x_i|^p \le C\left(\sum_{i=1}^{n} |x_i|^{(p-1)q}\right)^{1/q}$$

$$\Rightarrow \sum_{i=1}^{n} |x_i|^p \le C\left(\sum_{i=1}^{n} |x_i|^p\right)^{1/q}$$

$$\Rightarrow \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \le C.$$

Therefore, since $\left(\sum_{i=1}^{n}|x_i|^p\right)^{1/p}$ is uniformly bounded and monotone increasing in $n$ it follows that:

$$\|x\|_p^p = \lim_{n\to\infty}\left(\sum_{i=1}^{n}|x_i|^p\right) < \infty.$$

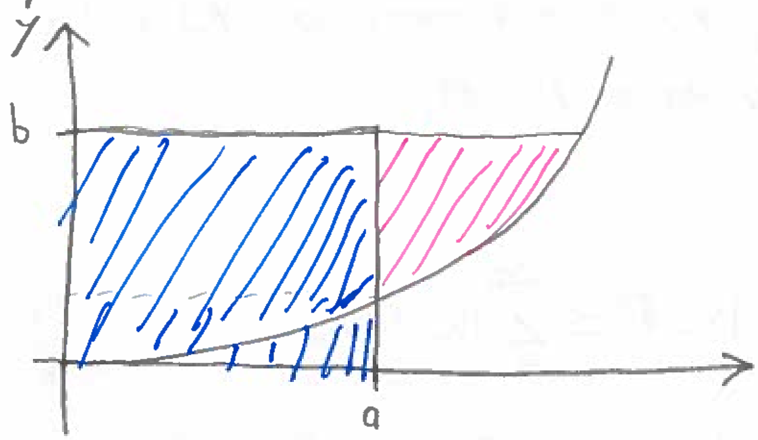
$\blacksquare$

# #2.

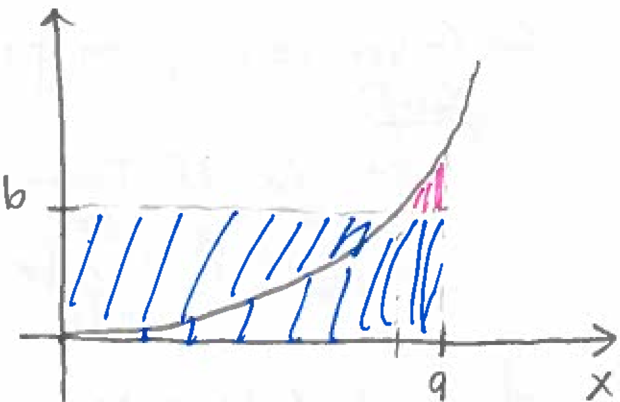
a.) Let $f$ be a continuous strictly increasing function on $[0,\infty)$ with $f(0)=0$. Prove that for $a,b>0$:

$$ab \le \int_0^a f(t)\,dt + \int_0^b f^{-1}(t)\,dt.$$

*proof:*




Case 1:




Case 2:

1. $$ab = \int_0^a f(x)\,dx + \int_0^{f(a)} f^{-1}(x)\,dx + a\cdot(b - f(a))$$
$$\le \int_0^a f(x)\,dx + \int_0^b f^{-1}(x)\,dx$$

2. $$ab = \int_0^{f^{-1}(b)} f(x)\,dx + \int_0^b f^{-1}(x)\,dx + b\cdot(a - f^{-1}(b))$$
$$\le \int_0^a f(x)\,dx + \int_0^b f^{-1}(x)\,dx.$$

b.) Prove that for all $a,b\ge 0$:

$$\exp(a) + (1+b)\ln(1+b) \ge (1+a)(1+b)$$

*proof:*

We apply Young's inequality with $f(x) = e^x - 1$.

$$\Rightarrow ab \le \int_0^a (e^x - 1)\,dx + \int_0^b \ln(x+1)\,dx$$
$$= e^a - a - 1 + (1+b)\ln(1+b) - b$$
$$\Rightarrow a + ab + b + 1 \le \exp(a) + (1+b)\ln(1+b)$$
$$\Rightarrow (1+a)(1+b) \le \exp(a) + (1+b)\ln(1+b).$$

$\blacksquare$

## #3.

a.) If $f, g \in C([0,1])$, show that

$$\int_0^1 |f(t)g(t)|\,dt \leq \|f\|_{L^p}\|g\|_{L^q}$$

with $1/p + 1/q = 1$, $1 < p < \infty$.

**proof:**

$$\frac{1}{\|f\|_p \cdot \|g\|_q}\int_0^1 |f(t)g(t)|\,dt = \int_0^1 \left|\frac{f(t)}{\|f\|_p}\right| \cdot \left|\frac{g(t)}{\|g\|_q}\right| dt$$

Therefore, by Young's inequality:

$$\left|\frac{f(t)}{\|f\|_p}\right| \cdot \left|\frac{g(t)}{\|g\|_q}\right| \leq \frac{1}{p}\frac{|f(t)|^p}{\|f\|_p^p} + \frac{1}{q}\frac{|g(t)|^q}{\|g\|_q^q}$$

$$\Rightarrow \int_0^1 \left|\frac{f(t)}{\|f\|_p}\right| \cdot \left|\frac{g(t)}{\|g\|_q}\right| dt \leq \frac{1}{p} + \frac{1}{q} = 1$$

$$\Rightarrow \int_0^1 |f(t)g(t)|\,dt \leq \|f\|_p \cdot \|g\|_q$$

b.) If $f, g \in C([0,1])$ and $1 < p < \infty$, show that

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p.$$

<u>Solution:</u>

$$|f(t)+g(t)|^p \leq |f(t)| \cdot |f(t)+g(t)|^{p-1} + |g(t)| \cdot |f(t)+g(t)|^{p-1}$$

Choose $q$ such that $1/p + 1/q = 1$. Therefore, by Hölder's inequality

$$\int_0^1 |f(t)| \cdot |f(t)+g(t)|^{p-1}\,dt \leq \left(\int_0^1 |f(t)|^p\,dt\right)^{1/p}\left(\int_0^1 |f(t)+g(t)|^p\right)^{1/q}$$

$$\int_0^1 |g(t)| \cdot |f(t)+g(t)|^{p-1}\,dt \leq \left(\int_0^1 |g(t)|^p\,dt\right)^{1/p}\left(\int_0^1 |f(t)+g(t)|^p\right)^{1/q}$$

$$\Rightarrow \frac{\int_0^1 |f(t)+g(t)|^p\,dt}{\left(\int_0^1 |f(t)+g(t)|^p\,dt\right)^{1/q}} \leq \|f\|_p + \|g\|_p$$

$$\Rightarrow \|f+g\|_p \leq \|f\|_p + \|g\|_p.$$

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# #4

a.) Let $f_n \in C([a,b])$ be a sequence of functions converging uniformly to a function $f$. Show that

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx.$$

*proof:*

$$\left| \int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx \right| = \left| \int_a^b (f_n(x) - f(x))\,dx \right|$$
$$\leq \int_a^b |f_n(x) - f(x)|\,dx$$
$$\leq \int_a^b \|f_n - f\|_\infty\,dx$$
$$= (b-a)\cdot\|f_n - f_\infty\|.$$

The result follows since $\|f_n - f\|_\infty \to 0$.

b.) Consider the space of continuously differentiable functions:

$$C^1([a,b]) = \{ f : [a,b] \to \mathbb{R} : f, f' \in C([a,b])$$

with the $C^1$ norm:

$$\|f\|_{C^1} = \|f\|_\infty + \|f'\|_\infty.$$

Prove that $C^1([a,b])$ is a Banach space.

*proof:*

Let $f_n$ be Cauchy in $C^1([a,b])$. Therefore, $f_n$ and $f_n'$ are Cauchy in $C([a,b])$ and hence converge uniformly to $f$ and $g$ respectively. Therefore,

$$\|f - f_n\|_{C^1} = \|f - f_n\|_\infty + \|g - f_n'\|_\infty \to 0.$$

To finish the proof we need to show that $f' = g$. By the Fundamental Theorem of Calculus:

$$f_n(t) - f_n(a) = \int_a^t f_n'(t)\,dt$$

Since $f_n' \to g$ uniformly it follows that

$$f(t) - f(a) = \int_a^t g(t)\,dt.$$

C.) Show that the space $C([a,b])$ with the $L^1$ norm $\|\cdot\|_{L^1}$ is incomplete. Show that if $f_n \to f$ with respect to $\|\cdot\|_{L^\infty}$ then $f_n \to f$ with respect to $\|\cdot\|_{L^1}$.

Solution:

On $[0,1]$ consider the following sequence of functions:

$$f_n(x) = \begin{cases} 0 & , \ 0 \le x \le \frac{1}{2} - \frac{1}{n} \\ \frac{n}{2}\left(x - \frac{1}{2} - \frac{1}{n}\right) & , \ \frac{1}{2} - \frac{1}{n} < x < \frac{1}{2} + \frac{1}{n} \\ 1 & , \ \frac{1}{2} + \frac{1}{n} \le x \le 1 \end{cases}$$

and define $f$ by:

$$f(x) = \begin{cases} 0 & , \ 0 \le x \le 1/2 \\ 1 & , \ 1/2 < x \le 1 \end{cases}$$

Computing, it follows that

$$\int_0^1 |f_n(x) - f(x)|\,dx = \frac{1}{2n^2}.$$

Therefore, $f_n \to f$ in $L^1$ but $f \notin L^1$.

Now, if $f_n \to f$ in $L^\infty$ then,

$$\int_a^b |f_n(x) - f(x)|\,dx \le \int_a^b \|f_n - f\|_\infty\,dx = (b-a)\|f_n - f\|_\infty$$

$\Rightarrow f_n \to f$ in $L^1$.

#5.

a.) For each of the following functions $f$, find the set of $p \in [1,\infty]$ for which $f \in L^p$ on the given interval.

i). $f(t) = e^{-t}$ on $[0,\infty)$.

Solution:

$1 \le p \le \infty$.

ii). $f(t) = 1/t$ on $[1,\infty)$.

Solution:

$1 < p \le \infty$.

iii). $f(t) = 1/\sqrt{t}$ on $(0,1]$.

Solution:

$$\int_0^1 t^{-p/2}\,dt = \begin{cases} \frac{2}{p} t^{1-p/2}\Big|_0^1 & \text{if } p \neq 2. \\ \ln(t)\Big|_0^1 & \text{if } p = 2 \end{cases}$$

$\Longrightarrow 1 \le p < 2$

iv). $f(t) = \ln(t)$, on $(0,3]$.

Solution:

$$\int_0^1 \ln(t)\,dt = t\ln(t) - t\Big|_0^1 = -1.$$

Assume $p$ is an integer. Therefore, integrating by parts:

$$\int_0^1 \ln(t)^p\,dt = t\ln(t)^p\Big|_0^1 - \int_0^1 p\ln(t)^{p-1}\,dt$$
$$= -\int_0^1 p\ln(t)^{p-1}\,dt.$$

It follows by induction that $\int_0^1 |\ln(t)|^p dt$ is finite. Since, $\int_0^1 |\ln(t)|^p dt$ is monotone in $p$ it follows that $\int_0^1 |\ln(t)|^p dt$ if finite for all $1 \le p < \infty$.

V.) $f(t) = 1/\sqrt{t}$ on $(0,\infty)$.

Solution:

$$\int_0^\infty t^{-p/2}\,dt = \begin{cases} \frac{2}{p} t^{1-p/2}\Big|_0^1 + \frac{2}{p} t^{1-p/2}\Big|_1^\infty & \text{if } p \neq 2 \\ \ln(t)\Big|_0^1 + \ln(t)\Big|_1^\infty & \text{if } p = 2 \end{cases}$$

Consequently, for all $p \in [1,\infty]$ $\int_0^\infty t^{-p/2}dt$ is divergent.

VI.) $f(t) = \begin{cases} t^{-1/2} & 0 < t \le 1 \\ 2t^{-2} & 1 < t < \infty \end{cases}$.

Solution:

$$\int_0^\infty |f(t)|^p\,dt = \int_0^1 t^{-p/2}\,dt + 2^p\int_1^\infty t^{-2p}\,dt$$

Now,

$$\int_1^\infty t^{-2p}\,dt = \frac{t^{-2p+1}}{1-2p}\Big|_1^\infty < \infty.$$

Therefore, by item (iii) it follows that $1 \le p < 2$.

c.) Is there a function $f$ on $(0,\infty)$ which belongs to $L_0^1$ but not $L_0^2$? Is there one that belongs to $L_0^2$ but not $L_0^1$?

Solution:

Let $f(t)=\begin{cases} 1/\sqrt{t} & 0<t\leq 1 \\ 0 & 1<t<\infty \end{cases}$, Then $f\in L_0^1$ but not $L_0^2$.

Let $g(t)=\begin{cases} 0 & 0<t\leq 1 \\ 1/t & 1<t<\infty \end{cases}$. Then $f\in L_0^2$ but not $L_0^1$.

d.) Is there a function $f$ on $(0,1)$ which belongs to $L_0^1$ but not to $L_0^2$? Is there one that belongs to $L_0^2$ but not $L_0^1$?

Solution:

Let $f(t)=1/\sqrt{t}$. Then $f\in L_0^1$.

Suppose $g\in L_0^2$. Then, by Hölder's inequality:

$$\int_0^1 |g(t)|\,dt \leq \left(\int_0^1 g(t)^2\,dt\right)^{1/2}\left(\int_0^1 1\,dt\right)^{1/2} = \|g\|_{L^2} < \infty.$$

Consequently $g\in L_0^1$.