Homework 7

Analysis

Due: March 19, 2018

1. **Displacement of a String:** A string of length L is clamped at its endpoints. It is under tension $T > 0$ and is subject to a transverse force per unit length $\tau(x) \geq 0$. The equilibrium displacement $u(x)$ then satisfies the following boundary value problem

$$Tu''(x) = -\tau(x),$$
$$u(0) = u(L) = 0.$$

Define the energy norm for this problem by:

$$\|f\|_E = \sqrt{\int_0^L [f'(x)]^2 \, dx}.$$

(a) Assuming there exists a smooth solution $u(x)$ to this problem, prove that

$$\|u\|_E = \sqrt{\frac{1}{T}\int_0^L \tau(x)u(x)\, dx}.$$

(b) Assuming there exists a smooth solution $u(x)$ to this problem, find upper bounds for the maximum displacement and also the total energy $\mathcal{E}[u] = T\|u\|_E^2/2$ in terms of the tension T, the length L of the string, and the root-mean applied forcing:

$$\bar{\tau} = \sqrt{\frac{1}{L}\int_0^L [\tau(x)]^2 \, dx}.$$

2. **Heat Equation:** Assume that the heat equation, and the associate boundary conditions

$$\frac{1}{2}\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t},$$
$$u(x,0) = u_0(x)$$
$$u(0,t) = u(L,t) = 0$$

has a smooth solution $u(x,t)$ for all smooth initial conditions $u_0(x)$.

(a) Show that the "spatial" L^2 and energy norms, decay as a function of time, and use this to prove the uniqueness of solutions of the heat equation.

(b) Show that there is a constant C such that

$$\|u(\cdot,t)\|_{L^2} \leq \|u_0\|_{L^2} e^{-Ct}.$$

(c) Show that the map $S_t : u_0 \mapsto u(x,t)$ is a continuous map from $L^2([0,L])$ into itself for all $t \geq 0$.

3. **Wave Equation:** Let $b > 0$. The wave equation with dissipation is given by

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + b\frac{\partial u}{\partial t},$$

with the initial boundary conditions:

$$\begin{aligned} u(x,0) &= u_0(x) \\ u_t(x,0) &= v_0(x) \\ u(0,t) &= u(L,t) = 0. \end{aligned}$$

Show the uniqueness of smooth solutions of this equation by constructing your own appropriate energy norm.

4. **Weierstrass Approximation Theorem:** Let

$$p_n(x) = \begin{cases} \left(1 - \frac{x^2}{4}\right)^n, & -2 \le x \le 2 \\ 0, & x < -2 \text{ or } x > 2 \end{cases},$$

$c_n = \int_{-\infty}^{\infty} p_n(x)\,dx$ and define $g_n(x)$ by $g_n(x) = \frac{1}{c_n} p_n(x)$.

(a) Let f be a continuous function on $\mathbb{R}$ compactly supported on $[-2,2]$. Prove that $f_n = g_n * f$ is a polynomial on $[-1,1]$.

(b) Prove that g_n is a sequence of averaging kernels.

(c) Prove that polynomials are dense in $C([-1,1])$.

(d) Prove that $C([-1,1])$ is separable, i.e. it is a dense countable subset.

Homework #7

#1

A string of length L is clamped at its endpoints. It is under tension $T>0$ and is subject to a transverse force per unit length $\tau(x)\geq 0$. The equilibrium displacement $u(x)$ then satisfies the following boundary value problem

$$Tu''(x) = -\tau(x),$$
$$u(0)=u(L)=0.$$

Define the energy norm for this problem by:

$$\|f\|_E = \left(\int_0^L f'(x)^2 dx\right)^{1/2}.$$

a.) Assuming there exists a smooth solution $u(x)$ to this problem, prove that

$$\|u\|_E = \sqrt{\frac{1}{T}\int_0^L \tau(x)u(x)dx}$$

Solution:

$$Tu''(x) = -\tau(x)$$
$$\Rightarrow Tu\cdot u'' = -u\tau(x)$$
$$\Rightarrow T\int_0^L u\cdot u''dx = -\int_0^L u\tau(x)dx$$
$$\Rightarrow T\cdot\left(u\cdot u'\Big|_0^L - \int_0^L u'(x)^2 dx\right) = -\int_0^L u\tau(x)dx$$
$$\Rightarrow \int_0^L u'(x)^2 dx = \frac{1}{T}\int_0^L u\tau(x)dx.$$
$$\Rightarrow \|u\|_E = \sqrt{\frac{1}{T}\int_0^L u\tau(x)dx}$$

b.) Assuming there exists a smooth solution $u(x)$ to this problem find upper bounds for the maximum displacement and also the total energy $\mathcal{E}[u] = T\|u\|_E^2/2$ in terms of the tension T, the length L of the string, and the root-mean applied forcing

$$\bar{\tau} = \sqrt{\frac{1}{L}\int_0^L \tau(x)^2 dx}.$$

Solution:

i.) By the Fundamental Theorem of Calculus:

$$|u(x)| = \left|\int_0^x u'(x)dx\right|$$
$$\leq \int_0^x |u'(x)|dx$$

$$\Rightarrow |u(x)| \le \int_0^L |u'(x)|\,dx$$
$$\le L^{1/2} \|u\|_E$$
$$= L^{1/2} \sqrt{\frac{1}{T}\int_0^L \tau(x) u(x)\,dx}$$
$$\le L^{1/2} \|u\|_\infty^{1/2} \cdot \sqrt{\frac{1}{T}\int_0^L \tau(x)\,dx}$$
$$\le L^{1/2} \|u\|_\infty^{1/2} \sqrt{\frac{1}{T}\left(\int_0^L \tau(x)^2 dx\right)^{1/2} \left(\int_0^L 1\,dx\right)^{1/2}}$$
$$= \frac{L^{1/2}\|u\|_\infty^{1/2}}{T^{1/2}} \cdot \sqrt{\frac{1}{L^2}\bar{\tau} \cdot L^{1/2}}$$
$$\Rightarrow \|u\|_\infty \le \frac{L^{1/2}\|u\|_\infty^{1/2}}{T^{1/2}} \cdot \sqrt{\frac{1}{L^2}\bar{\tau} \cdot L^{1/2}}$$
$$\Rightarrow \|u\|_\infty^{1/2} \le \frac{L^{1/2}}{T^{1/2}} \sqrt{\frac{1}{L^2}\bar{\tau} L^{1/2}}$$
$$\Rightarrow \|u\|_\infty \le \frac{L}{T} \cdot \frac{L^{1/2}}{L^2} \bar{\tau} = \frac{1}{T L^{1/2}} \bar{\tau}.$$

ii) $$\frac{T\|u\|_E^2}{2} = \frac{\int_0^L \tau(x) u(x)\,dx}{2} \le \frac{\|u\|_\infty}{2} \int_0^L \tau(x)\,dx$$
$$\Rightarrow \mathcal{E}[u] \le \|u\|_\infty \left(\int_0^L \tau^2(x)\,dx\right)^{1/2} L^{1/2}$$
$$\le \frac{1}{T L^{1/2}} \cdot L^{1/2} \bar{\tau} \left(\int_0^L \tau^2(x)\,dx\right)^{1/2}$$
$$= \frac{L^{1/2}}{T} \bar{\tau}^2.$$

∎

#2.

Assume that the heat equation, and the associated boundary conditions

$$\frac{1}{2} u_{xx} = u_t$$
$$u(x,0) = u_0(x)$$
$$u(0,t) = u(L,t)$$

has a smooth solution $u(x,t)$ for all smooth initial conditions $u_0(x)$.

a.) Show that the "spatial" L^2 and energy norms decay as a function of time, and use this to prove uniqueness of solutions to the heat equation.

Solution:

i) $$\|u(x,t)\|_{L^2}\frac{d}{dt}\|u(x,t)\|_{L^2}=\frac{d}{dt}\|u(x,t)\|_{L^2}^2$$
$$=\frac{d}{dt}\int_0^L u(x,t)^2dx$$
$$=2\int_0^L u(x,t)\frac{d}{dt}u(x,t)\,dx$$
$$=\int_0^L u(x,t)\,u_{xx}(x,t)dx$$
$$=-\int_0^L u_x(x,t)\,u_x(x,t)dx$$
$$=-\int_0^L u_x(x,t)^2dx$$
$$\le 0.$$

Moreover, since $-\int_0^L u_x(x,t)^2dx\le -C\int_0^L u(x,t)^2dx$ it follows that:
$$\|u(x,t)\|_{L^2}\frac{d}{dt}\|u(x,t)\|_{L^2}\le -C\|u(x,t)\|_{L^2}^2$$
$$\Longrightarrow \frac{d}{dt}\|u(x,t)\|_{L^2}\le -C\|u(x,t)\|_{L^2}\le 0. \qquad (*).$$

ii) Multiplying the heat equation by u it follows that:
$$\frac{1}{2}u\cdot u_{xx}=u\cdot u_t$$
$$\Longrightarrow \int_0^L\frac{1}{2}u\cdot u_{xx}\,dx=\int_0^L u\cdot u_t\,dx$$
$$\Longrightarrow -\frac{1}{2}\int_0^L u_x^2dx=\int_0^L u\cdot u_t\,dx$$
$$\Longrightarrow -\frac{1}{2}\frac{d}{dt}\int_0^L u_x^2dx=\int_0^L(u_t^2+u\cdot u_{tt})dx$$
$$=\int_0^L(u_t^2+u\cdot u_{txx})dx$$
$$=\int_0^L(u_t^2-\frac{1}{2}u_x u_{tx})dx$$
$$=\int_0^L(u_t^2-\frac{1}{4}\frac{d}{dt}u_x^2)dx$$

$$\Longrightarrow \frac{d}{dt}\int_0^L u_x^2dx\le -4\int_0^L u_t^2\le 0$$

iii). To prove uniqueness let u, v be solutions. Then, $u-v$ solves the heat equation and thus by $(*)'$

$$\frac{1}{2}\frac{d}{dt}\|u-v\|_{L^2} \leq 0$$

However, $\|u(0,x)-v(0,x)\|_{L^2} = \|u_0-u_0\|_{L^2} = 0$. Therefore, since $\|u-v\|$ is monotonically decreasing in t and bounded below by 0 it follows that for all t $\|u-v\|_{L^2} = 0$.

b.) By $(*)$ it follows from Gronwall's inequality that

$$\|u(x,t)\|_{L^2} \leq \|u_0\|_{L^2} e^{-ct}. \quad (**)$$

c.) Show that the map $S_t : u_0 \longmapsto u(x,t)$ is a continuous map from $L^2([0,L])$ into itself for all $t \geq 0$.

<u>Solution:</u>

Let $u_0^{(n)} \in L^2$ satisfy $u_0^{(n)} \to u_0$ then by $(**)$

$$\|S_t u_0^{(n)} - S_t u_0\|_{L^2} \leq \|u_0^{(n)} - u_0\|_{L^2} e^{-ct} \longrightarrow 0.$$

<u>#3</u>

Let $b>0$. The wave equation with dissipation is given by:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + b\frac{\partial u}{\partial t}$$

With the initial boundary conditions:

$$u(x,0) = u_0(x)$$
$$u_t(x,0) = v_0(x)$$
$$u(0,t) = u(L,t) = 0.$$

Show the uniqueness of smooth solutions of this equation by constructing your own appropriate energy norm.

<u>Solution:</u>

Define $\|u\|_E$ by:

$$\|u\|_E^2 = \frac{1}{2}\int_0^L \left(u_t^2(x,t) + u_x^2(x,t)\right) dx.$$

Differentiating, it follows that

$$2\|u\|_E \frac{d}{dt}\|u\|_E = \int_0^L \left(u_t u_{tt} + u_x u_{xt}\right) dx$$

$$\Rightarrow 2\cdot\|u\|_E \frac{d}{dt}\|u\|_E = \int_0^L [u_t(u_{xx} - bu_t) + u_x u_{xt}]\,dx$$

$$= \cancel{u_t u_x}\Big|_0^L + \int_0^L (-u_{tx}u_x - bu_t^2 + u_x u_{xt})\,dx$$

$$= -\int_0^L -bu_t^2\,dx$$

$$\leq 0.$$

Therefore, if u, v solve the wave equation with dissipation then

$$\|u(x,t) - v(x,t)\|_E \leq 0.$$

However, $\|u(x,0) - v(x,0)\|_E = 0$ and thus for all t!

$$\|u(x,t) - v(x,t)\|_E = 0.$$

#4.

Let

$$p_n(x) = \begin{cases} (1 - x^2/4)^n, & -2 \leq x \leq 2 \\ 0, & x < -2 \text{ or } x > 2 \end{cases}$$

$C_n = \int_{-\infty}^{\infty} p_n(x)\,dx$ and define $g_n(x)$ by $g_n(x) = \frac{1}{C_n} p_n(x)$.

a.) Let f be a continuous function on $\mathbb{R}$ compactly supported on $[-2,2]$. Prove that $f_n = g_n * f$ is a polynomial on $[-1,1]$.

proof:

$$f_n = g_n * f = \int_{-\infty}^{\infty} g_n(x-y) f(y)\,dy$$

$$= \frac{1}{C_n}\int_{-\infty}^{\infty}\left(1 - \frac{(x-y)^2}{4}\right)^n f(y)\,dy$$

$$= \frac{1}{C_n}\int_{-2}^{2}\sum_{k=0}^{2n} d_k x^k y^{2n-k} f(y)\,dy$$

$$= \frac{1}{C_n}\sum_{k=0}^{2n} x^k \int_{-2}^{2} d_k y^{2n-k} f(y)\,dy$$

$$= \sum_{k=0}^{2n} a_k x^k,$$

where a_k is a constant.

d.) Prove that $C([-1,1])$ is seperable.

proof:

The result follows since g_n is an averaging kernel and we can consider polynomials with rational coefficients.

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