Homework 9

Analysis

Due: April 09, 2018

1. If T is a contraction mapping for $c < 1$, show that we can estimate the distance of the fixed point x^* from the initial point x_0 of the fixed point iteration by

$$d(x^*, x_0) \leq \frac{1}{1-c} d(x_1, x_0).$$

2. Using the contraction mapping theorem, show that the following differential equation has a unique solution

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{1}{4}\left(1 + \sin^2(x)\right) \\ x(0) = 1 \end{cases} .$$

3. The following integral equation for $f : [-a, a] \mapsto \mathbb{R}$ arises in a model of the motion of gas particles on a line:

$$f(x) = 1 + \frac{1}{\pi} \int_{-a}^{a} \frac{1}{1 + (x-y)^2} f(y)\, dy,$$

for $-a \leq x \leq a$. Prove that this equation has a unique bounded, continuous solution for every $0 < a < \infty$. Prove that the solution is nonnegative. What can you say if $a = \infty$?

4. **Logistic Equation:** The logistic map $T : [0, 1] \mapsto [0, 1]$ is defined by

$$x_{n+1} = Tx_n = rx_n(1 - x_n),$$

where $r > 0$.

(a) Show that for $0 < r < 1$ there is a single fixed point x^* for this problem. Prove that if $0 < r < 1$ then for all $x_0 \in [0, 1]$

$$\lim_{n \to \infty} T^n x_0 = x^*.$$

(b) For what values of r will there be exactly two fixed points for this problem. For what range of r will at least one of these fixed points be stable.

(c) A period two orbit is a point x^* satisfying $T^2 x^* = x^*$. For what ranges of r will there exist period two orbits. For what ranges of r will there exist a stable period two orbit?

Homework #9

#1.

If T is a contraction mapping for $c<1$, show that we can estimate the distance of the fixed point x^* from the initial point x_0 of the fixed point iteration by

$$d(x^*, x_0) \leq \frac{1}{1-c} d(x_1, x_0).$$

Solution:

By continuity it follows that:

$$\begin{aligned}
d(x^*, x_0) &= \lim_{n\to\infty} d(x_n, x_0) \\
&\leq \lim_{n\to\infty} [d(x_n, x_1) + d(x_1, x_0)] \\
&= \lim_{n\to\infty} [d(T(x_{n-1}), T(x_0)) + d(x_1, x_0)] \\
&\leq \lim_{n\to\infty} [c\, d(x_{n-1}, x_0) + d(x_1, x_0)] \\
&\leq \lim_{n\to\infty} [c(d(x_{n-1}, x_1) + d(x_1, x_0)) + d(x_1, x_0)].
\end{aligned}$$

Continuing inductively it follows that:

$$\begin{aligned}
d(x^*, x_0) &\leq \lim_{n\to\infty} [d(x_1, x_0)(1 + c + c^2 + \ldots + c^{n-1})] \\
&= \frac{1}{1-c} d(x_1, x_0).
\end{aligned}$$

$\blacksquare$

#2.

Using the contraction mapping theorem, show that the following differential equation has a unique solution.

$$\begin{cases} \frac{dx}{dt} = \frac{1}{4}(1+\sin^2(x)) \\ x(0) = 1 \end{cases}$$

Solution:

The above equation is equivalent to the following integral equation:

$$x(t) = 1 + \frac{1}{4}\int_0^t (1+\sin^2(x(s)))ds. \quad (*)$$

Define $T: \mathcal{C}([0,t]) \to \mathcal{C}([0,t])$ by

$$T(x(t)) = 1 + \frac{1}{4}\int_0^t (1+\sin^2(x(s)))ds.$$

Consequently, solutions to $(*)$ correspond to fixed points of T.
For $0<t<1$ it follows that

$$|T(x(t)) - T(y(t))| = \frac{1}{4}\left|\int_0^t [\sin^2(x(s)) - \sin^2(y(s))]ds\right|$$
$$\leq \frac{1}{4}\int_0^t |\sin^2(x(s)) - \sin^2(y(s))|ds.$$

By the mean-value theorem it follows that

$$|\sin^2(x(s)) - \sin^2(y(s))| \leq 2|\cos(c)\sin(c)|\cdot|x(s)-y(s)|$$
$$\leq 2\cdot|x(s)-y(s)|.$$

Therefore,

$$|T(x(t)) - T(y(t))| \leq \frac{1}{4}\int_0^t 2|x(s)-y(s)|ds$$
$$\leq \frac{1}{2}\int_0^t \|x-y\|_\infty ds$$
$$= \frac{1}{2}\|x-y\|_\infty \int_0^t ds$$
$$< \frac{1}{2}\|x-y\|_\infty$$

$$\Rightarrow \|Tx - Ty\| < \frac{1}{2}\|x-y\|_\infty.$$

Therefore, by the contraction mapping theorem T has a unique fixed point $x_1^* \in \mathcal{C}([0,1])$.

We can use the same argument to show the existence of a unique solution on the interval $[1,2]$ for the following differential equation:

$$\begin{cases} \frac{dx}{dt} = \frac{1}{4}(1+\sin^2(x)), \\ x(1) = x_1^*(1) \end{cases}$$

Continuing inductively gives a unique solution on all of $\mathbb{R}$. ∎

#3.

The following integral equation for $f:[-a,a]\to\mathbb{R}$ arises in a model of the motion of gas particles on a line:

$$f(x) = 1 + \frac{1}{\pi}\int_{-a}^{a} \frac{1}{1+(x-y)^2} f(y)\,dy,$$

for $-a \le x \le a$. Prove that this equation has a unique bounded, continuous solution for every $0<a<\infty$. Prove that the solution is nonnegative. What can you say if $a=\infty$?

Solution:

Define $T_a : \mathcal{C}([-a,a]) \to \mathcal{C}([-a,a])$ by

$$T_a(f(x)) = 1 + \frac{1}{\pi}\int_{-a}^{a} \frac{1}{1+(x-y)^2} f(y)\,dy.$$

Therefore,

$$\begin{aligned} |T_a(f(x)) - T_a(g(x))| &= \frac{1}{\pi}\left|\int_{-a}^{a} \frac{1}{1+(x-y)^2}(f(y)-g(y))\,dy\right| \\ &\le \frac{1}{\pi}\int_{-a}^{a} \frac{|f(y)-g(y)|}{1+(x-y)^2}\,dy \\ &\le \frac{\|f-g\|_\infty}{\pi}\int_{-a}^{a} \frac{1}{1+(x-y)^2}\,dy \\ &< \frac{\|f-g\|_\infty}{\pi}\int_{-\infty}^{\infty} \frac{1}{1+(x-y)^2}\,dy \\ &= \|f-g\|_\infty. \end{aligned}$$

Consequently, if $a \ne \infty$ it follows from the contraction mapping theorem that T_a has a fixed point $f^* \in \mathcal{C}([-a,a])$ i.e. a solution to the integral equation. From completeness of $\mathcal{C}([-a,a])$, f^* is continuous on $[-a,a]$ and hence bounded.

Now, let f_m^* denote the minimum value of f^* and assume for contradiction that $f_m^* < 0$. Then, there exists $c \in [-a, a]$ such that

$$f_m^* = 1 + \frac{1}{\pi}\int_{-a}^{a} \frac{1}{1+(c-y)^2} f(y)\,dy$$
$$> 1 + \frac{1}{\pi}\int_{-a}^{a} \frac{1}{1+(c-y)^2} f_m^*\,dy$$
$$> 1 + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{1+(c-y)^2} f_m^*\,dy,$$

where the last inequality follows from the assumption that $f_m^* < 0$. Consequently,

$$f_m^* > 1 + f_m^*$$
$$\Rightarrow 0 > 1.$$

This is a contradiction and thus $f_m^* > 0$.

Finally, if $a = \infty$ we do not get a strict contraction and thus the contraction mapping theorem does not apply.

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#4 Logistic Equation

The logistic map $T: [0,1] \to [0,1]$ is defined by

$$x_{n+1} = Tx_n = r x_n (1 - x_n),$$

where $r > 0$.

a.) Show that for $0 < r < 1$ there is a single fixed point x^* for this problem. Prove that if $0 < r < 1$ then for all $x_0 \in [0,1]$:

$$\lim_{n \to \infty} T^n x_0 = x^*.$$

Solution:

A fixed point satisfies

$$x^* = r x^* (1 - x^*)$$

which implies

$$x^* = 0 \quad \text{or} \quad 1 = r(1 - x^*)$$
$$\Rightarrow x^* = 0 \quad \text{or} \quad 1 - \frac{1}{r} = x^*$$

The second fixed point exists in $[0,1]$ if and only if $r > 1$.

Furthermore, if $r<1$ it follows that

$$|Tx-0| = |rx(1-x)| < |x(1-x)| < |x| = |x-0|.$$

Consequently, T is globally contracting to 0.

b.) For what values of r will there be exactly two fixed points for this problem. For what range of r will at least one of the fixed points be stable.

Solution:

It follows from the previous calculation that if $r>1$ there are two fixed points. Let

$$f(x) = rx(1-x).$$

Calculating it follows that

$$f'(x) = r(1-2x)$$

$$\Rightarrow f'(0) = r \text{ and } f'\left(1-\frac{1}{r}\right) = 2-r.$$

Consequently, 0 is stable if $0<r<1$ and $1-\frac{1}{r}$ is stable if $1<r<3$.

C.) A period two orbit is a point x^* satisfying $T^2x^* = x^*$. For what ranges of r will there exist period two orbits. For what ranges of r will there exist a stable period two orbit?

Solution:

A period two orbit satisfies:

$$x^* = T(T(x^*)) = S(x^*),$$

where $S(x)$ is defined by

$$S(x) = T(rx(1-x)) = r[r(x(1-x)](1-[r(x(1-x))])$$

$$\Rightarrow S(x) = -r^3x^4 + 2r^3x^3 - (r^2+r^3)x^2 + r^2x$$

Thus, we need to find roots of the equation:

$$r^3x^4 - 2r^3x^3 + (r^2+r^3)x^2 - r^2x + x = 0$$

$$\Rightarrow x(1-r+rx)(1+r-rx-r^2x+r^2x^2) = 0$$

Consequently, the non-trivial period two orbit is given by:

$$x_{\pm}^* = \frac{1+r \pm \sqrt{(1+r)(r-3)}}{2}$$

which exists if $r > 3$. Stability is a mess. However,

$$S'(x) = T'(T(x))T'(x)$$

$$\Rightarrow S'(x_+^*) = T'(x_-^*)T'(x_+^*)$$

$$= 4 + 2r - r^2$$

$$\Rightarrow |S'(x_+^*)| < 1 \Rightarrow |4 + 2r - r^2| < 1$$

$$\Rightarrow \quad 3 < r < 1 + \sqrt{6}.$$

That is, the period two orbit is stable if $3 < r < 1 + \sqrt{6}$ $\blacksquare$