Lecture #1: Review of 611 via examples

What is analysis?

- Rigorous foundation of calculus (narrow view)
- Rigorous approximations (numerical view)
- Mathematical study of limiting processes (mature view)
- Applications - probability, differential equations, physics, dynamical systems, applied math.

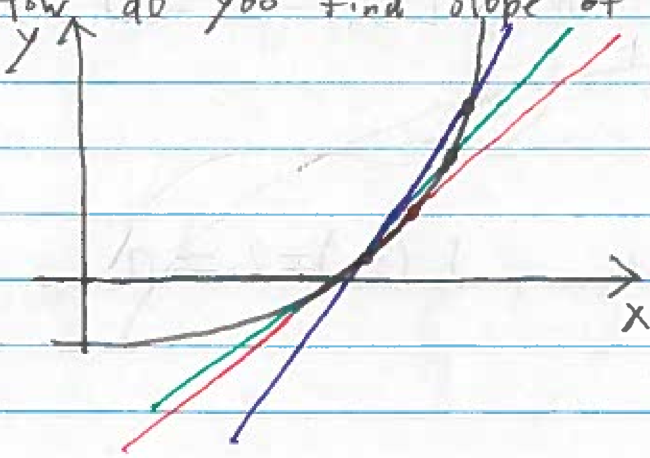
My view:

- Start with hard problem.
- Approximate hard problem with simple problem that can be solved exactly.
- Take limit.

* Analysis is the necessary glue needed to ensure this process works.

example 1:

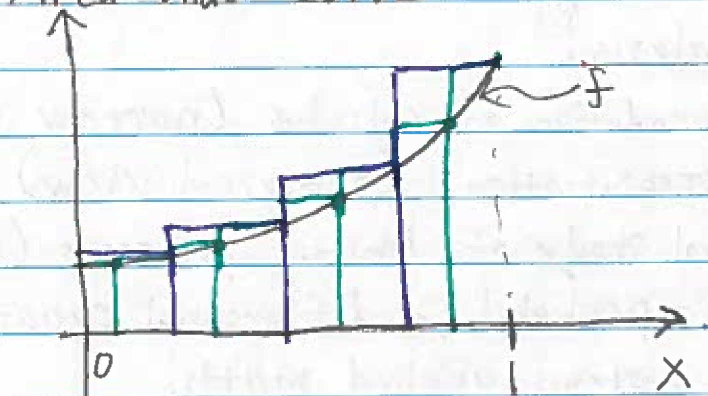
How do you find slope of tangent line



$f'(x)$ = limit of secant lines.

example 2:

Area under curve



$f_n \to$ Function with n rectangles.

$f_4 =$ •

$f_8 =$ •

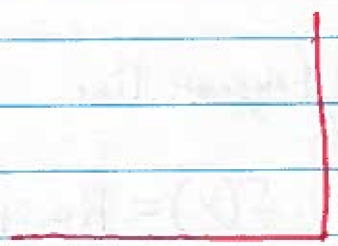
$\lim_{n\to\infty} A(f_n) = \int_0^1 f(x)\,dx$, A is the area function.

example 3:

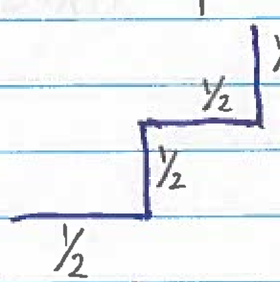
$\sqrt{2} = 2$.

proof:

Let L denote the length of a curve

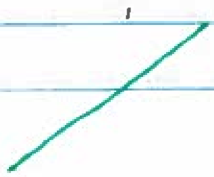


$= \alpha_1$, $L(\alpha_1) = 2 = a_1$



$\frac{1}{2} = \alpha_2$, $L(\alpha_2) = 2 = a_2$

⋮ ⋮



$= \alpha_\infty$ $L(\alpha_\infty) = \sqrt{2}$

$$\sqrt{2} = L(\alpha_\infty) = L\left(\lim_{n\to\infty} \alpha_n\right) = \lim_{n\to\infty} L(\alpha_n) = \lim_{n\to\infty} 2 = 2.$$

The proof is obviously wrong.

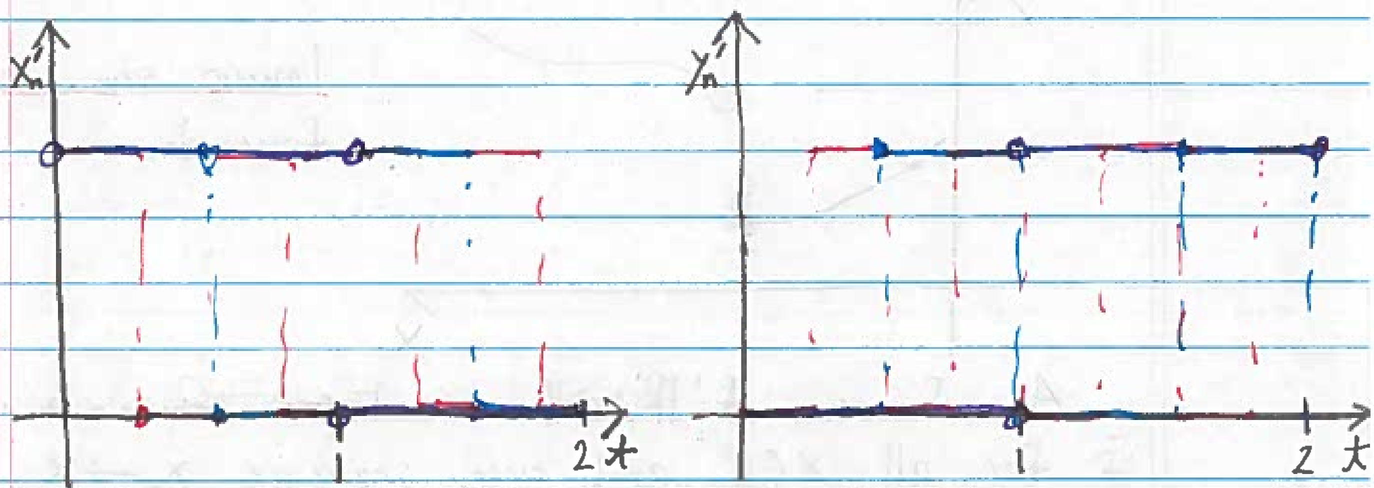
$$L\left(\lim_{n\to\infty} \alpha_n\right) \neq \lim_{n\to\infty} L(\alpha_n).$$

We only knew it was wrong because we know the answer already.

Lets take a deeper look at this example.

$\alpha_n(t) = (x_n(t), y_n(t))$, (parametric form of curve).

$$\Longrightarrow L(\alpha_n(t)) = \int_0^2 \sqrt{x'(t)^2 + y'(t)^2}\, dt$$



$\alpha_1' = ★$
$\alpha_2' = ★$
$\alpha_3' = ★$

The functions x_n', y_n' converge to infinite wiggly function

The strange fact is

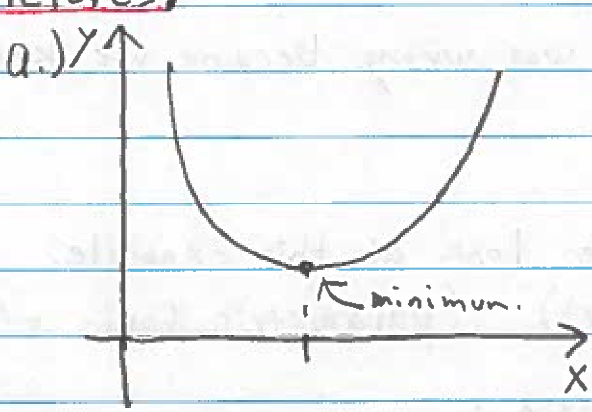
$$\lim_{t\to\infty} x_n(t) = \tfrac{1}{2}t, \quad \lim_{t\to\infty} y_n(t) = \tfrac{1}{2}t$$

But: $\lim_{t\to\infty} x_n'(t) \neq \tfrac{1}{2}$, $\lim_{t\to\infty} y_n'(t) \neq \tfrac{1}{2}$ (limit does not exist) example 4:

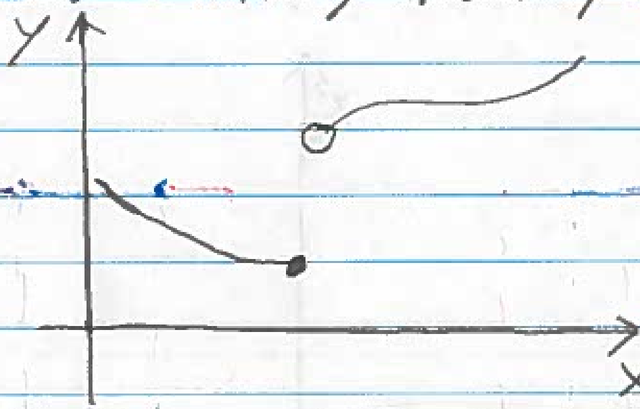
Consider a function $f: \mathbb{R} \mapsto \mathbb{R}$. Under what conditions does f have a minimum?

Pictures:

a.)



Is continuity necessary? No.



lower semicontinuous function

A function $f: \mathbb{R} \mapsto \mathbb{R}$ is lower semicontinuous if for all $x \in \mathbb{R}$ and every sequence $x_n \to x$, we have

$$f(x) \leq \liminf_{n \to \infty} f(x_n)$$

Recall:

1. Let $b_n = \sup\{x_k : k \geq n\}$.

$$\limsup_{n \to \infty} x_n = \lim_{n \to \infty} b_n = \lim_{n \to \infty} \sup\{x_k : k \geq n\}$$

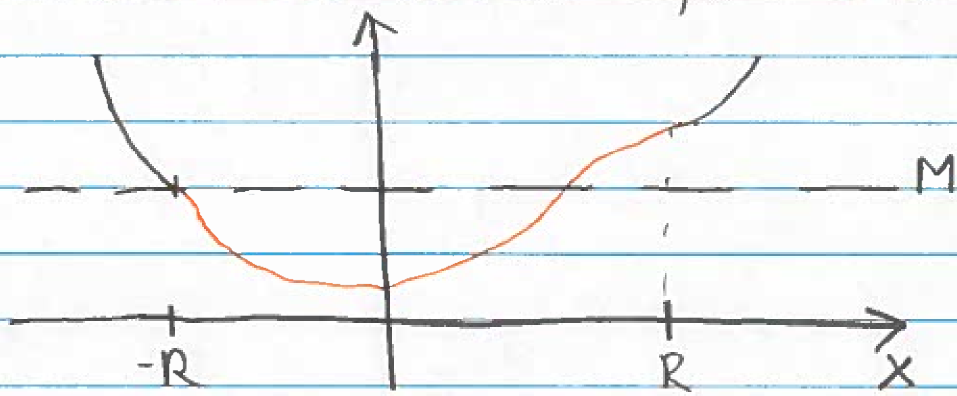
2. Let $a_n = \inf\{x_k : k \geq n\}$

$$\liminf_{n \to \infty} x_n = \lim_{n \to \infty} a_n = \lim_{n \to \infty} \sup\{x_k : k \geq n\}.$$

Definition – A function $f: \mathbb{R} \to \mathbb{R}$ is coercive if

$$\lim_{|x| \to \infty} f(x) = \infty$$

* **Explicitely**: For any $M > 0$, there exists $R > 0$ such that $|x| > R$ implies $f(x) \geq M$.



Theorem – If $f: \mathbb{R} \mapsto \mathbb{R}$ is lower semicontinuous and coercive, then there exists x^* such that for all $x \in \mathbb{R}$:

$$f(x^*) \leq f(x).$$

proof:

Let x_n be a mimizing sequence of f. I.e.

$$\lim_{n \to \infty} f(x_n) = \inf_{x \in \mathbb{R}} f(x).$$

Since f is coercive there exists $R > 0$ such that for all $n \in \mathbb{N}$, $|x_n| < R$. Therefore, by Bolzano-Weirstrass there exists $x^* \in [-R, R]$ and a subsequence x_{n_k} such that

$$\lim_{k \to \infty} x_{n_k} = x^*.$$

Therefore,

$$f(x^*) \leq \liminf_{k \to \infty} f(x_{n_k}) = \lim_{n \to \infty} f(x_n) = \inf_{x \in \mathbb{R}} f(x)$$

$$\Longrightarrow f(x^*) = \inf f(x)$$