Lecture 10: Completion of a Metric Space

Equivalence Relations:

Definition: We say a binary relation on X is an *equivalence relation* if:

- Reflexivity: For all $x \in X$, $x \sim x$
- Symmetry: For all $x, y \in X$ if $x \sim y$ then $y \sim x$
- Transitivity: For all $x, y, z \in X$ if $x \sim y$ and $y \sim z$ then $x \sim z$

Definition: Let $\sim$ be an equivalence relation on X. The *equivalence class* of $x \in X$ relative to an equivalence relation $\sim$ is the set

$$[x] = \{y \in X : y \sim x\}.$$

Example:

Let $X = \mathbb{Z} \times \mathbb{Z}$. Define,

$$(a,b) \sim (c,d) \text{ if } ad = bc \ (\text{i.e. } \tfrac{a}{b} = \tfrac{c}{d}).$$

$[(a,b)] = \mathbb{Q}$.

Example:

Let $X = \{\{x_n \in \mathbb{Q} : x_n \text{ is Cauchy}\}$. Define,

$x_n \sim y_n$

if $\lim_{n\to\infty} |x_n - y_n| = 0$. Then, $[x_n] = \mathbb{R}$.

Abstract Completions

Let (X, d) be a metric space. Let

$$Y = \{ x_n \in X : x_n \text{ is Cauchy} \}$$

If $x_n, y_n \in Y$ we say $x_n \sim y_n$ if $\lim_{n\to\infty} d(x_n, y_n) = 0$

Lemma - The relation $\sim$ is an equivalence relation.

proof:

Let $x_n, y_n, z_n \in Y$.

1. $d(x_n, x_n) = 0 \Rightarrow \lim_{n\to\infty} d(x_n, x_n) = 0 \Rightarrow x_n \sim x_n$.
2. If $x_n \sim y_n$ then

$$\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(y_n, x_n) = 0$$

$$\Rightarrow y_n \sim x_n.$$

3. Suppose $x_n \sim y_n$ and $y_n \sim z_n$. Then,

$$\limsup_{n\to\infty} d(x_n, z_n) \leq \limsup_{n\to\infty} (d(x_n, y_n) + d(y_n, z_n)) = 0.$$

$\blacksquare$

Definition - Define $\tilde{X}$ as the quotient of Y by $\sim$.

Lemma - If $x_n, y_n, \bar{x}_n, \bar{y}_n \in Y$ and $x_n \sim \bar{x}_n, y_n \sim \bar{y}_n$.
then

$$\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(\bar{x}, \bar{y}_n)$$

proof:

$$d(x_n, y_n) \leq d(x_n, \bar{x}_n) + d(\bar{x}_n, \bar{y}_n) + d(y_n, \bar{y}_n)$$

$$\Rightarrow \lim_{n\to\infty} d(x_n, y_n) \leq d(\bar{x}_n, \bar{y}_n)$$

$$d(\bar{x}_n, \bar{y}_n) \leq d(\bar{x}_n, x_n) + d(x_n, y_n) + d(y_n, \bar{y}_n)$$

$$\Rightarrow \lim_{n\to\infty} d(\bar{x}_n, \bar{y}_n) \leq \lim_{n\to\infty} d(x_n, y_n)$$

Lemma- The function $\tilde{d}: \tilde{X} \times \tilde{X} \to \mathbb{R}$ given by $\tilde{d}([p_n],[q_n]) = \lim_{n\to\infty} d(p_n, q_n)$ is a metric on $\tilde{X}$.

Proposition- Let $i: X \to \tilde{X}$ be given by $i(x) = [x]$. Then i is an isometry, i.e. for every $x, y \in X$,

$$\tilde{d}(i(x), i(y)) = d(x,y)$$

proof:

$$\tilde{d}(i(x), i(y)) = \lim_{n\to\infty} d(i(x)_n, i(y)_n) = \lim_{n\to\infty} d(x,y) = d(x,y).$$

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Proposition- The image $i(X)$ is dense in $\tilde{X}$.

proof:

Let $[x_n] \in \tilde{X}$ and let $\varepsilon > 0$. Since $x_n \in X$ is Cauchy there exists $N \in \mathbb{N}$ such that $m, n \geq N \Rightarrow d(x_m, x_n) < \varepsilon$. Therefore, for $k \geq N$ if we let $x = x_k$ then $i(x) = (x_k, x_k, \ldots)$ and thus

$$\tilde{d}([x_n], i(x)) < \varepsilon.$$

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Lemma- Suppose that Y is a dense subset of a metric space X. If every Cauchy sequence in Y converges to an element of X, then X is complete.

proof:

Suppose x_n is Cauchy in X. Since Y is dense there exists $y_n \in Y$ so that $d(x_n, y_n) < 1/n$. Therefore,

$$d(y_n, y_m) \leq 1/n + d(x_n, x_m) + 1/m$$

$$\Rightarrow y_n \text{ is Cauchy in } Y \text{ and therefore } y_n \to y \in X.$$

Finally,

$$d(x_n, y) \leq d(x_n, y_n) + d(y_n, y)$$

$$\Rightarrow x_n \to y.$$

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Theorem - The metric space $(\tilde{X}, \tilde{d})$ is complete.

~~proof~~:

We know $i(X)$ is dense in $\tilde{X}$. Let $i(x_n)$ be Cauchy in $i(X)$. Because i is an isometry, this implies x_n is Cauchy in X. Consequently, there exists $c \in \tilde{X}$ such that $c = [x_n]$. We want to show $i(x_n) \to c$ in $\tilde{X}$. Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ so that $n, m \geq N$ implies $d(x_n, x_m) < \varepsilon$. Then, for $n \geq N$

$$\tilde{d}(i(x_n), [x_m]) = \tilde{d}((x_n, x_n, \dots), (x_1, x_2, \dots))$$
$$= \lim_{m \to \infty} d(x_n, x_m)$$
$$\leq \varepsilon.$$

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Theorem - Let (X, d) be a metric space. Then there exists a complete metric space $(\tilde{X}, \tilde{d})$, called the completion of X, and a natural embedding $i: X \to \tilde{X}$ such that for all $x, y \in X$, $\tilde{d}(i(x), i(y)) = d(x, y)$. Moreover, $i(X)$ is dense in $\tilde{X}$.