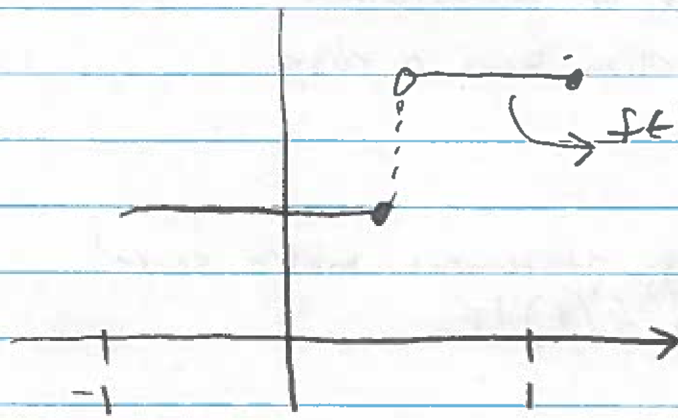
Lecture 11: Completions of Function Spaces

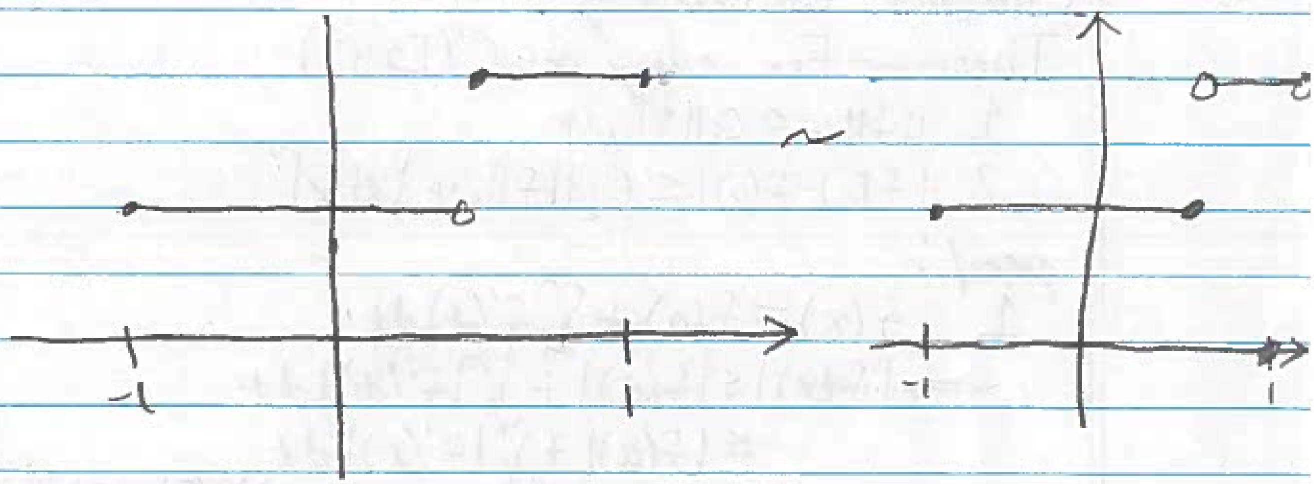
Definition - $L^p(K)$ is the completion of $\mathcal{C}(K)$ with respect to the L^p metric.

- Are elements of $L^p(K)$ functions?
- If $f \in L^p(K)$ what does $f(0)$ mean?



$\hookrightarrow f \in L^p([-1,1])$, but $f \in \mathcal{C}(K)$.

- What does $\sim$ mean in this context?



Definition -

$W^{1,p}([0,1])$ = the completion of $C^1([0,1])$ with respect to the norm:

$$\|f\|_{W^{1,p}} = \left(\int_0^1 |f(x)|^p dx + \int_0^1 |f'(x)|^p dx\right)^{1/p}.$$

- What kind of functions are in $W^{1,p}$
- Can $W^{1,p}$ functions be discontinuous.
- Can a $W^{1,p}$ function have a cusp.

example:

Does the following statement make sense!

$$f(x) = f(0) + \int_0^x f'(t)\,dt$$

if $f \in W^{1,2}(0,1)$?

Compact Embeddings.

Theorem - For every $f \in C^1([a,b])$

1. $\|f\|_\infty \leq C_1 \|f\|_{W^{1,p}}$
2. $|f(x) - f(y)| \leq C_2 \|f\|_{W^{1,p}} |x-y|^{1/p}$

proof:

1. $f(x) = f(a) + \int_a^x f'(t)\,dt$

$$\Rightarrow |f(x)| \leq |f(a)| + \int_a^x |f'(t)|\,dt$$
$$\leq |f(a)| + \int_a^b |f'(t)|\,dt$$
$$\leq |f(a)| + \left(\int_a^b |f'(t)|^p dt\right)^{1/p} \left(\int_a^b 1^q dt\right)^{1/q}$$
$$= |f(a)| + \|f'\|_{L^p} (b-a)^{1/q}$$

We also have

$$f(a) = f(x) - \int_a^x f'(t)\,dt$$
$$\Rightarrow |f(a)| \leq |f(x)| + \int_a^x |f'(t)|\,dt$$
$$\leq |f(x)| + \|f'\|_{L^p} \cdot (b-a)^{1/q}$$

$$\Rightarrow \int_a^b |f(a)|\,dx \le \int_a^b |f(x)|\,dx + \|f'\|_{L^p} \cdot (b-a)^{q+q/q}$$
$$\le \|f\|_{L^p} \cdot (b-a)^{1/q} + \|f'\|_{L^p} \cdot (b-a)^{q+1/q}.$$

Therefore,

$$|f(x)| \le \|f\|_{L^p} \cdot (b-a)^{1/q} + \|f'\|_{L^p} \cdot [(b-a)^{q+1/q} + (b-a)^{1/q}]$$
$$\Rightarrow \|f(x)\|_{L^\infty} \le C\|f\|_{W^{1,2}}.$$

2. $f(y) - f(x) = \int_x^y f'(t)\,dt$

$$\Rightarrow |f(y)-f(x)| \le \int_x^y |f'(t)|\,dt$$
$$\le \left(\int_x^y |f'(t)|^p\,dt\right)^{1/p} \cdot \left(\int_x^y dt\right)^{1/q}$$
$$\le \|f'\|_{L^p} \cdot |y-x|^{1/q}$$
$$\le \|f\|_{W^{1,p}} \cdot |y-x|^{1/q}$$

Theorem - If f_n is is a bounded sequence in $(C'([0,1], \|\cdot\|_{W^{1,p}})$ then f_n has a convergent subsequence.

~~proof~~

Item 1 above implies f_n is bounded in $\|\cdot\|_\infty$. Item 2 implies equicontinuity. The result follows from Arzela-Ascoli.

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