# Lecture 13: Linear Operators

**Definition** - A linear map or linear operator between normed linear spaces $X, Y$, is a function $T: X \to Y$ such that

$$T(\lambda x + \nu y) = \lambda Tx + \nu Ty, \quad \text{for } \lambda, \nu \in \mathbb{R},\ x, y \in X.$$

**Definition** - A linear operator is bounded if there is a constant $M \geq 0$ such that

$$\|Tx\| \leq M\|x\| \quad \text{for all } x \in X.$$

If no such constanst exists we say $T$ is unbounded.
The operator norm is defined by

$$\|T\|_{op} = \|T\| = \inf\{M : \|Tx\| \leq M\|x\|\}.$$

$$\Rightarrow \|T\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{\|x\| \leq 1} \|Tx\| = \sup_{\|x\|=1} \|Tx\|.$$

**Example:**
Let $X = C^1([0,1], \|\cdot\|_\infty)$, $Y = C^0([0,1], \|\cdot\|_\infty)$. Define

$$T: X \to Y \quad \text{by} \quad Tf = \frac{df}{dx}.$$

Then, $T$ is unbounded since if

$$f_n = e^{nx}$$

it follows that

$$\|f_n\|_\infty = 1, \quad \|Tf_n\|_\infty = \|ne^{nx}\|_\infty = n$$

$$\Rightarrow \|T\| = \sup_{\|x\|=1} \|Tx\| = \infty.$$

Example:

Let $X = (C([0,1]), \|\cdot\|_\infty)$. Define $K: X \to X$ by
$$Kf(x) = \int_0^x f(y)\,dy.$$

$$\Rightarrow \|Kf\|_\infty = \sup_x \left|\int_0^x f(y)\,dy\right|$$
$$\leq \sup_x \int_0^x |f(y)|\,dy$$
$$\leq \int_0^1 |f(y)|\,dy$$
$$\leq \|f\|_\infty$$

Therefore, $K$ is bounded and $\|K\| \leq 1$. In fact,
$$\|K(1)\|_\infty = \sup_x \left|\int_0^x 1\,dx\right| = 1.$$

$$\Rightarrow \|K\|_\infty = 1.$$

Definition - A linear operator $T: X \to \mathbb{R}$ is called a linear functional.

Example:

Suppose $1 < p < \infty$ and $q \in \mathbb{R}$ satisfies $\frac{1}{p} + \frac{1}{q} = 1$.
Let $y \in \ell^q$ and define $T: \ell^p \to \mathbb{R}$ by:
$$Tx = \sum_{i=1}^\infty y_i x_i$$

$$\Rightarrow |Tx| = \left|\sum_{i=1}^\infty y_i x_i\right| \leq \sum_{i=1}^\infty |y_i x_i| \leq \|y\|_{\ell^q} \|x\|_{\ell^p}$$

$$\Rightarrow \|T\| \leq \|y\|_{\ell^q}$$

To get equality set $x_i = |y_i|^{q-1} \cdot \operatorname{sgn}(y_i)$.
$$\Rightarrow \|T\| = \|y\|_{\ell^q}.$$

Theorem - A linear map is bounded if and only if it is continuous.

~~proof~~

1. If $T: \mathfrak{X} \to \mathfrak{Y}$ is bounded, then

$$\|Tx - Ty\| = \|T(x-y)\| \leq M \cdot \|x-y\|$$

$\Rightarrow T$ is continuous.

2. Suppose $T$ is continuous at 0. By linearity $T(0)=0$. Therefore, there exists $\delta > 0$ such that

$$\|Tx\| \leq 1 \quad \text{when} \quad \|x\| \leq \delta.$$

For any $x$, define $\tilde{x}$ by

$$\tilde{x} = \delta \frac{x}{\|x\|}$$

$$\Rightarrow \|\tilde{x}\| \leq \delta$$

$$\Rightarrow \|Tx\| = \frac{\|x\|}{\delta} \|T\tilde{x}\| \leq \frac{\|x\|}{\delta}.$$

Definition - Let $\mathfrak{X}, \mathfrak{Y}$ be normed linear spaces, then

$$B(\mathfrak{X}, \mathfrak{Y}) = \{L : \mathfrak{X} \to \mathfrak{Y} \,;\, L \text{ is bounded and linear}\}.$$

$*$ $B(\mathfrak{X}, \mathfrak{Y})$ is a normed linear space with $\|\cdot\|_{op}$.

**Theorem** - If $\mathbb{Y}$ is complete, then $\mathcal{B}(\mathbb{X},\mathbb{Y})$ is complete.

*proof:*

1. Let $L_n \in \mathcal{B}(\mathbb{X},\mathbb{Y})$ be a Cauchy sequence. Let $x \in \mathbb{X}$. Then

$$\|L_n(x) - L_m(x)\|_{\mathbb{Y}} = \|L_n(\mathbb{X}-\mathbb{Y})\|_{\mathbb{Y}} \leq \|L_n - L_m\|_{op} \cdot \|x\|_{\mathbb{X}}$$

$$\Rightarrow L_n(x) \text{ is Cauchy in } \mathbb{Y}.$$

Define

$$L(x) = \lim_{n\to\infty} L_n(x)$$

2. Since $L_n$ is Cauchy it follows that there exists $K>0$ such that

$$\|L_n\| \leq K.$$

Therefore, for $x \in \mathbb{X}$ with $\|x\|=1$ it follows that

$$\|L_n(x)\|_{\mathbb{Y}} = \lim_{n\to\infty} \|L_n(x)\| \leq K$$

$$\Rightarrow \|L_n\|_{\mathbb{Y}} \leq K.$$

3. Also, if $\|x\|_{\mathbb{X}}=1$ it follows that

$$\|(L-L_n)x\|_{\mathbb{Y}} \leq \|(L-L_m)(x)\|_{\mathbb{Y}} + \|(L_m-L_n)(x)\|_{\mathbb{Y}}$$

There exists $N \in \mathbb{N}$ such that $m,n \geq N$ implies

$$\|(L_m-L_n)x\|_{\mathbb{Y}} \leq \varepsilon$$

$$\Rightarrow \|(L-L_n)x\|_{\mathbb{Y}} \leq \|(L-L_m)(x)\|_{\mathbb{Y}} + \varepsilon$$

Taking $m\to\infty$ it follows that for $n \geq N$:

$$\|(L-L_n)x\|_{\mathbb{Y}} < \varepsilon$$

$$\Rightarrow L_n \to L \text{ in the operator norm.}$$