Lecture 14: Calculus of Variations and Weak Solutions

Example:

Let $A = \{f \in C^1([0,1]) : f(0) = f(1) = 0\}$. Define $T : A \to \mathbb{R}$ by

$$T[f] = \int_0^1 [(f(x)-1)^2 + f'(x)^2]\,dx.$$

Find f that minimizs T.

Suppose f minimizes T. Let $\eta \in C_c^\infty([0,1];\mathbb{R})$. Define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(h) = T[f + \underbrace{h\eta}_{\text{perturbation}}]$$

* $f + h\eta \in A$

Now, $g(0)$ is a local minimumum $\Rightarrow g'(0) = 0$.

$$\frac{dg}{dh} = \frac{d}{dh} T[f + h\eta] = \frac{d}{dh}\left[\int_0^1 (f(x) + h\eta(x) - 1)^2\,dx\right]$$

$$+ \frac{d}{dh}\left[\int_0^1 (f'(x) + h\eta'(x))^2\,dx\right]$$

$$\Rightarrow \frac{dg}{dh} = 2\int_0^1 (f(x) + h\eta(x) - 1)\cdot\eta(x)\,dx$$

$$+ 2\int_0^1 (f'(x) + h\eta'(x))\cdot\eta'(x)\,dx.$$

$$\Rightarrow \boxed{\left.\frac{dg}{dh}\right|_{h=0} = 2\int_0^1 (f(x) - 1)\eta(x)\,dx + 2\int_0^1 f'(x)\eta'(x)\,dx}$$

We say that f is a <u>weak solution to the Euler-Lagrange equations</u> if for all $\eta \in C_c^{\infty}$

$$2\int_0^1 (f(x)-1)\eta(x)\,dx + 2\int_0^1 f'(x)\eta'(x)\,dx = 0$$

If f is second differentiable we can integrate by parts:

$$\Rightarrow 2\int_0^1 (f(x)-1)\eta(x)\,dx + 2f'(x)\cancel{\eta(x)}\Big|_0^1 - 2\int_0^1 f''(x)\eta(x)\,dx$$

$$\Rightarrow 2\int_0^1 [(f(x)-1)-f''(x)]\,\eta(x)\,dx$$

$$\Rightarrow \boxed{\begin{array}{l} f''(x) = f(x)-1 \\ f(0)=f(1)=0 \end{array}} \rightarrow \text{strong form of Euler-Lagrange equations.}$$

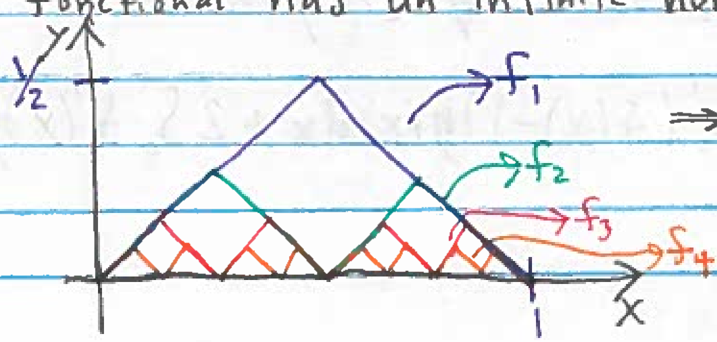
A function f satisfying the above equation is known as a <u>strong solution to the Euler-Lagrange equations</u>.

Example:

Let $A = \{ f \in W^{1,4}([0,1]) : f(0)=f(1)=0 \}$. Define $I : A \to \mathbb{R}$ by

$$I[f] = \int_0^1 (f'(x)^2-1)^2\,dx.$$

This functional has an infinite number of minimizers.



$\Rightarrow I[f_n]=0.$

Example:

Let $A = \{f \in W^{1,4}([0,1]) : f(0) = f(1) = 0\}$. Define $I : A \to \mathbb{R}$ by

$$\ast \quad I[f] = \int_0^1 f(x)^2 dx + \int_0^1 (f'(x)^2 - 1)^2 dx.$$

This functional has no minimizer.

1. $I[f] > 0$ since f cannot satisfy both $f \equiv 0$ and $f' = \pm 1$.
2. Define f_n as in the previous example. Then,
$$I[f_n] = \int_0^1 f_n(x)^2 dx \leq 2^{n-1} \cdot \left(\frac{1}{2^n}\right)^2 = \frac{1}{2^{n+1}}.$$
Therefore, $I[f_n] \to 0$. Consequently,
$$\inf_{f \in A} I[f] = 0.$$

By items 1 and 2, I has no minimum.

Variational Derivative:

Let $A = \{f \in C^2[a,b] : f(a) = f(b) = 0\}$. Define $I : A \to \mathbb{R}$ by

$$I[f] = \int_a^b L(x, f(x), f'(x)) dx.$$

$L : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a smooth function called the Lagrangian.

Let $g(h) = I[f + h\eta]$, for $\eta \in C_c^\infty([a,b])$. Then,

$$g'(h) = \frac{d}{dh} \int_a^b L(x, f(x) + h\eta, f'(x) + h\eta'(x)) dx$$
$$= \int_a^b [L_f(x, f(x) + h\eta(x))\eta(x) + L_{f'}(x, f'(x) + h\eta'(x))\eta'(x)] dx$$

$$\Rightarrow g'(0) = \int_a^b [L_f(x, f(x), f'(x))\eta(x) + L_{f'}(x, f(x), f'(x))\eta'(x)] dx$$
$$= \int_a^b [L_f(x, f(x), f'(x))\eta(x) - \frac{d}{dx} L_{f'}(x, f(x), f'(x))]\eta(x) dx$$

The strong form of the Euler-Lagrange equations are therefore:

$$\star\; L_f(x,f(x),f'(x)) - \frac{d}{dx}L_{f'}(x,f(x),f'(x)) = 0. \;\star$$

The variational derivative or functional derivative of $I = \int_a^b L(x,f(x),f'(x))dx$ at C^2 function f is the function:

$$\frac{\delta I}{\delta f} = L_f - \frac{d}{dx}L_{f'}$$

The Euler-Lagrange equation for I is then

$$\frac{\delta I}{\delta f} = 0.$$

Local Linearization:

$$\frac{d}{d\varepsilon} I[f+\varepsilon y] = \int_0^1 \frac{\delta I}{\delta f} y\, dx$$

$\Rightarrow$ For a fixed f this defines a linear operator.

$*$ If $f \in L^p$ then

$$\int_0^1 \frac{\delta I}{\delta f} y\, dx$$

is bounded if $y \in L^q$. (Follows from Hölders).

The linearization of I about f is the operator:

$$I[f+\varepsilon y] = I[f] + \varepsilon \int_0^1 \frac{\delta I}{\delta f} y\, dx + o(\varepsilon^2)$$

This follows since

$$\lim_{\varepsilon\to 0} \frac{I[f+\varepsilon y] - I[f]}{\varepsilon} = \int_0^1 \frac{\delta I}{\delta f} y\, dx.$$