Lecture 15: Weak Convergence

Definition – Suppose that $1 \leq p < \infty$. A sequence f_n converges weakly to f in $L^p([a,b])$ written $f_n \rightharpoonup f$, if

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} f_n g \, dx = \int_{-\infty}^{\infty} f(x) g(x) \, dx,$$

for every $g \in L^q$, $\left(\frac{1}{p} + \frac{1}{q} = 1\right)$.

Example:

a.) Let $f_n(x) = \sin(n\pi x)$. Then $f_n \rightharpoonup 0$ on $[0,1]$ in L^2.

b.) Let $f \in L^2([0,1])$. Define $f_n(x) = f(x-n)$. Then,
$f_n \rightharpoonup 0$ in L^2.

c.) Let $f \in L^2$. Define $f_n(x)$ by
$$f_n(x) = n^{1/2} f(nx).$$

Construction of weak topology

A topology τ on X is a collection of subsets satisfying

a.) $\emptyset, X \in \tau$

b.) If $G_\alpha \in \tau$ for $\alpha \in A$, the $\bigcup_{\alpha \in A} G_\alpha \in \tau$

c.) If $G_i \in \tau$ for $i = 1, \ldots, n$, then $\bigcap_{i=1}^{n} G_i \in \tau$

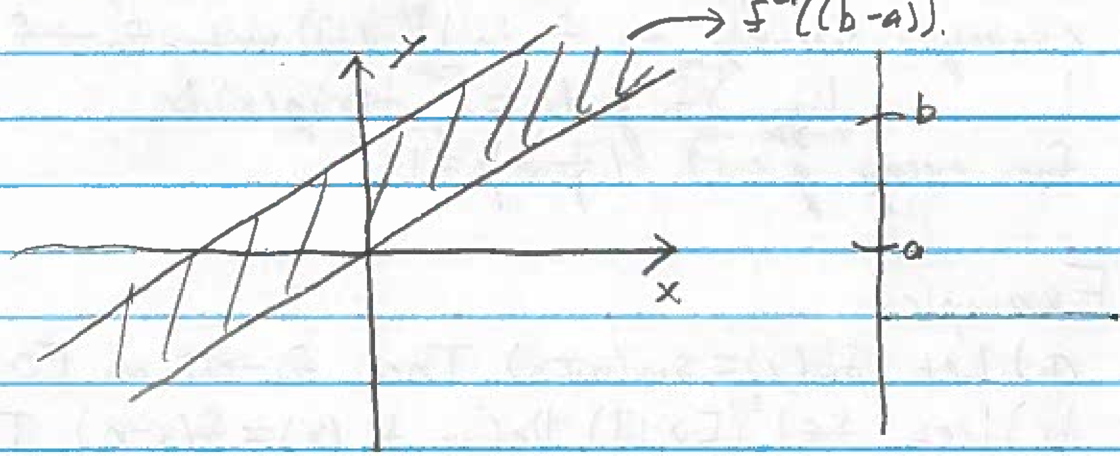
Def-

A function $f : X \to Y$ between topological spaces (X, τ) and (Y, S) is continuous if for all $G \in S$, $f^{-1}(G) \in \tau$.

Goal:

Let X be a linear space. Construct a topology in which all linear functionals are continuous.

Define $G \in \tau$ if $G = f^{-1}(\text{open set in } \mathbb{R})$ for some linear functional f.



$L\vec{x} = a_1x_1 + a_2x_2$ $\qquad \mathbb{R}$

$\Rightarrow a < a_1x_1 + a_2x_2 < b$

* In infinite dimensions this topology is weaker than the metric topology.

Definition - Let X be a linear space and X^* its dual a sequence $x_n \in X$ converges weakly to x (written $x_n \rightharpoonup x$) if for all $\varphi \in X^*$

$$\lim_{n\to\infty} \varphi(x_n) = \varphi(x).$$

Examples:

1. $\ell^{2*} = \ell^2$. If $x^{(n)} \in \ell^2$ satisfies $x^{(n)} \rightharpoonup x$ then $\lim_{n\to\infty} \sum_{i=1}^{\infty} x_i^{(n)} y_i = \sum_{i=1}^{\infty} x_i y_i$ for all $y \in \ell^2$.