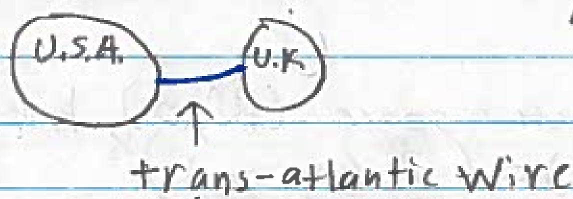
Lecture 2: Metric Spaces

RR - pg. 1-29
HN - pg. 1-17

Idea of metric space:

How do we extend calculus to more abstract mathematical objects?

1. $r, s \in \mathbb{R}$, $|r-s|$ is distance between numbers.
2. When are two linear equations the same:
   $Ax = b \quad \tilde{A}x = b$
   (Important question in numerical linear algebra)
   Need useful conclusions:
   $\|A - \tilde{A}\|$ small implies solutions are small
3. Transmission of a signal $F_{sent}(t)$, $F_{received}$

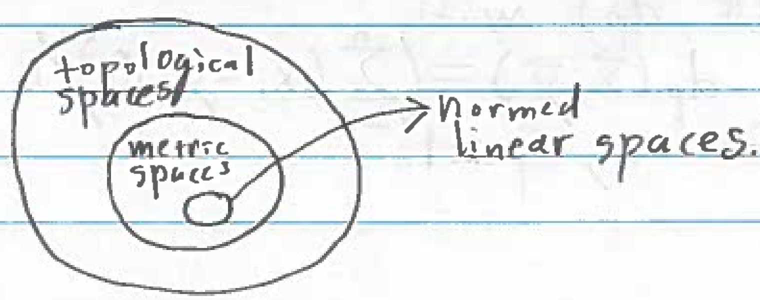


$$\text{Error} = \| F_{sent} - F_{recieved} \|$$

initial condition   solution to P.D.E.

First Goal of Course

1. Metric Space - Quantifies distance
2. Normed Linear Space - Quantifies size and distance



Metric Space:

Recall for $x, y, z \in \mathbb{R}$

1. $|x-y| \geq 0$ and $|x-y| = 0 \iff x = y$
2. $|x-y| = |y-x|$
3. $|x-y| \leq |x-z| + |z-y|$

All of the above properties generalize the notion of distance.

Definition - A metric space is a pair (X, d), where X is a set and $d: X \times X \mapsto [0, \infty)$, called a metric on X, is a function satisfying

(nondegeneracy) 1. $d(x,y) = 0$ if and only if $x = y$

(symmetry) 2. $d(x,y) = d(y,x)$, for all $x, y \in X$

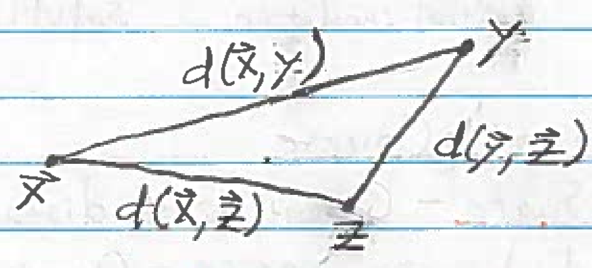
(triangle inequality) 3. $d(x,y) \leq d(x,z) + d(z,y)$, for all $x, y, z \in X$.

examples:

1. Euclidean metric space $(\mathbb{R}^n, d)$ has metric

$$d(\vec{x}, \vec{y}) = \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{1/2},$$

$$\vec{x} = (x_1, \ldots, x_n),\ \vec{y} = (y_1, \ldots, y_n).$$

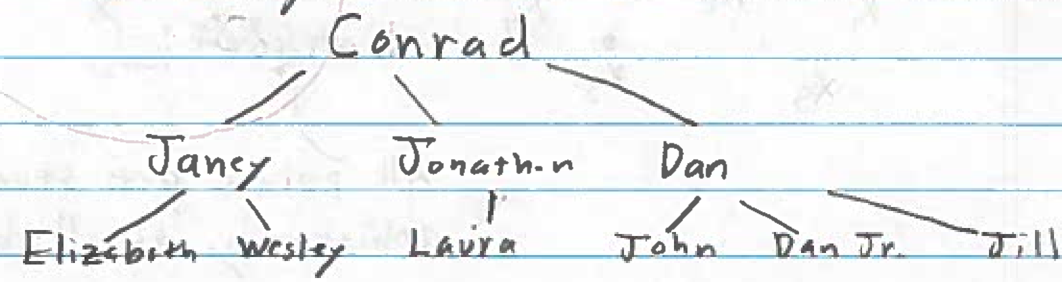


2. $(\mathbb{R}^n, d_p)$ with

$$d_p(\vec{x}, \vec{y}) = \left(\sum_{i=1}^{n} (x_i - y_i)^p\right)^{1/p}$$

for any $p \geq 1$.

3. $(\mathcal{F}, d)$, metric space of family tree with
$d =$ lineage distance to common ancestor



$d(\text{Elizabeth}, \text{Wesley}) = 1$

$d(\text{Laura}, \text{Janey}) = 2$

triangle inequality:

$d(x,y) \leq d(x,z) + d(z,y)$

<u>Convergence:</u>

<u>Definition</u> - Let (X, d) be a metric space. A sequence $x_n \in X$ converges to $x \in X$ if

$$\lim_{n\to\infty} d(x_n, x) = 0.$$

We also denote $x_n \to x$.

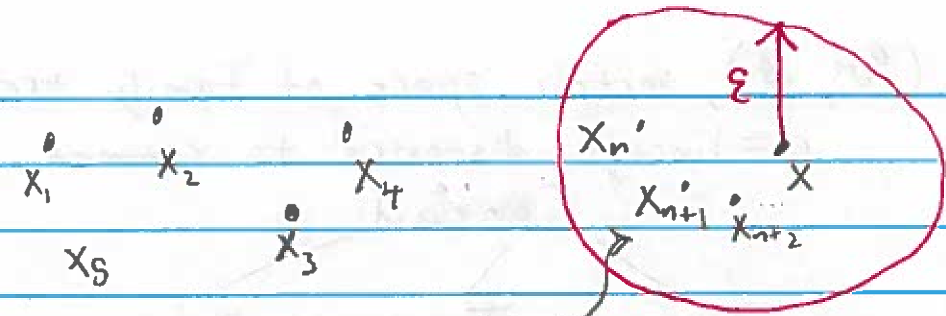
<u>Definition</u> - The set $B_r(x_0)$ defined by

$$B_r(x_0) = \{x : d(x, x_0) < r\}.$$

is called the open ball of radius r.

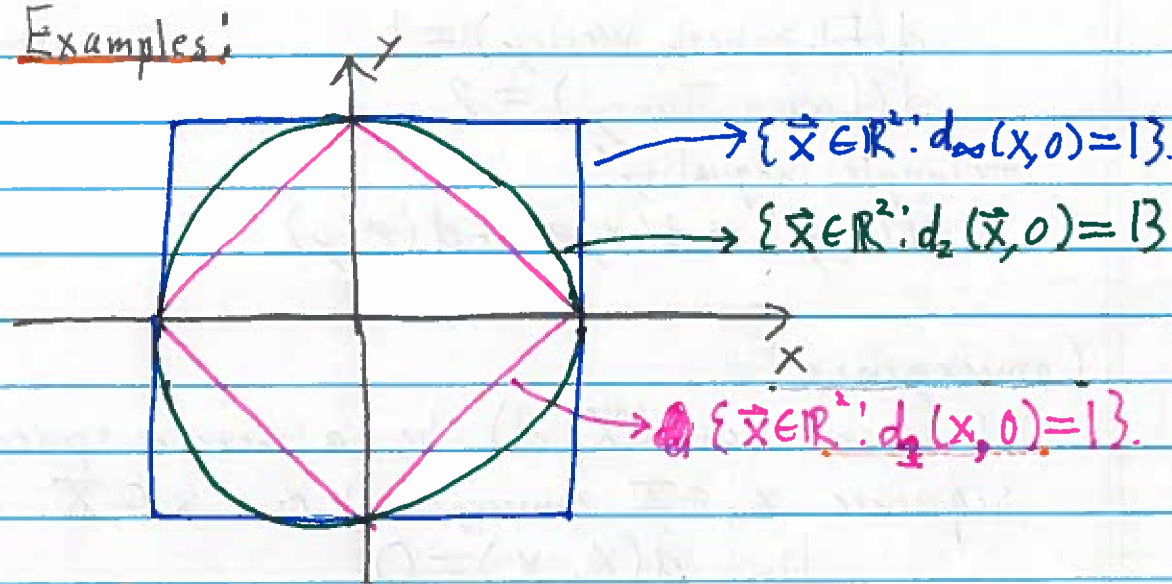
<u>Definition</u> - Let (X, d) be a metric space. A sequence $x_n \in X$ converges to $x \in X$ if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N$ implies

$$x_n \in B_\varepsilon(x_0).$$



All points get stuck inside of arbitrarily small balls.

Examples:



"Unit balls"

Note: Because of the geometry of the d_p unit balls if $x_n \xrightarrow{d_q} x$ in $\mathbb{R}^2$ for some $1 \le q \le \infty$ then $x_n \xrightarrow{d_p} x$ for all $p \in [1, \infty]$.

Non-Example:

Consider $(\mathbb{R}, d)$ with

$$d_{\frac{1}{2}}(x, y) = (|x_1 - y_1|^{\frac{1}{2}} + |x_2 - y_2|^{\frac{1}{2}})^2$$

Is this a metric space? No.

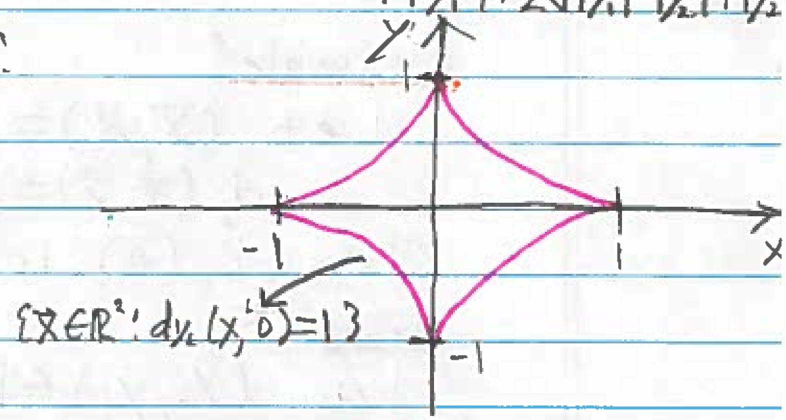
If this is a metric space we must have

$$d_{1/2}(\vec{x},\vec{y}) \le d_{1/2}(\vec{x},0) + d_{1/2}(0,\vec{y})$$

$$\Rightarrow |x_1-y_1|^{1/2} + 2\sqrt{|x_1-y_1|\cdot|x_2-y_2|} + |x_2-y_2|^{1/2} \le |x_1| + 2\sqrt{|x_1|\cdot|x_2|} + |x_2|$$
$$+ |y_1| + 2\sqrt{|y_1|\cdot|y_2|} + |y_2|$$

Make special choice:

$$x_1 = 1,\ y_1 = 0$$
$$x_2 = 0,\ y_2 = 1$$
$$\Rightarrow 1 + 2 + 1 \le 1 + 1$$
$$\Rightarrow \quad 2 \le 0$$



Unit ball is not convex.

Cauchy sequences / compactness

Definition – Let (X, d) be a metric space.
A sequence $x_n \in X$ is Cauchy if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $m,n \ge N \Rightarrow d(x_m, x_n) < \varepsilon$.

Definition – Let (X, d) be a metric space and $x_n \in X$.
A subsequence of x_n is a sequence x_{n_k} so that for all k

$$x_n = x_{n_k}$$

for some $n \in \mathbb{N}$, $n_{k+1} > n_k$ for each k.

* $k \to n_k$ is a strickly increasing function.

example:

$$x_n = 1/n$$

$x_{n_k} = 1/k^2$ is a subsequence of x_n,

$$k \to k^2$$

$$x_n = \{1, 1/2, 1/3, 1/4, 1/5, \ldots, 1/9, \ldots\}$$
$$x_{n_k} = \{1, 1/4, 1/9, \ldots\}$$

Definition - A set $K \subset X$ is compact if every sequence in K has a convergent subsequence that converges to an element in K.

example:

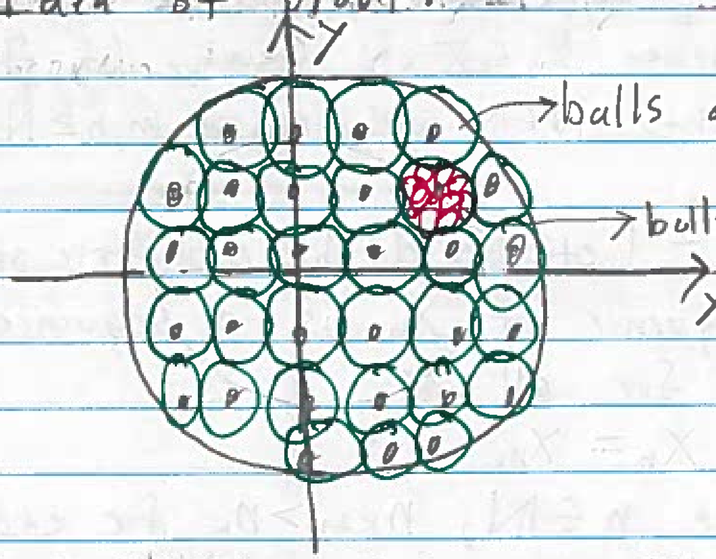
Let $(X, d) = (\mathbb{R}^2, d_2)$, where
$$d_2(\vec{x}, \vec{y}) = (|x_1 - y_1|^2 + |x_2 - y_2|^2)^{1/2}$$

Claim $B_1(0)$ is compact in this metric space.

~~proof~~:

Let $(x_n, y_n) \in B_1(0)$. Therefore, for all $n \in \mathbb{N}$:
$$x_n^2 + y_n^2 \leq 1.$$

Idea of proof:



1. Let $B_{1/10}(\vec{z}_i)$ denote covering of $B_1(0)$ by balls of radius $1/10$

By pigion hole principle there exists j such that infinitely many $(x_n, y_n) \in B_{1/10}(\vec{z}_j)$. Let
$$n_1 = \min\{n \in \mathbb{N} : (x_n, y_n) \in B_{1/10}(\vec{z}_j)\}.$$

2. Let $B_{1/100}(\vec{z}_i)$ denote covering of $B_{1/10}(\vec{z}_j)$ by balls of radius $1/100$. By pigeon hole principle there exists j such that infinitely many $(x_n, y_n) \in B_{1/100}(\vec{z}_j)$

Let $n_2 = \min\{n \in \mathbb{N} : n > n_1 \text{ and } (x_n, y_n) \in B_{1/100}(\vec{z}_1)\}$.

3. Continuing inductively, we construct a subsequence $\vec{X}_{n_j}$ satisfying $d_2((x_{n_j}, y_{n_j}), \vec{z}_1) < 1/10^j$.
Therefore, by triangle inequality
$$d((x_{n_j}, y_{n_j}), (x_{n_{j+1}}, y_{n_{j+1}})) \leq 2/10^j = 2 \cdot 10^{-j}.$$

4. By homework #1, (x_{n_j}, y_{n_j}) is a Cauchy sequence and therefore has a limit.

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