Lecture 3: Normed Linear Spaces.

Definition - A linear space $\mathbb{X}$ over $\mathbb{R}$ is a set on which

1. $\mathbb{X}$ is a commutative group with respect to an operation $+$.
   a.) $x, y \in \mathbb{X} \Rightarrow x+y = y+x$
   b.) $x, y, z \in \mathbb{X} \Rightarrow (x+y)+z = x+(y+z)$.
   c.) There exists $0 \in \mathbb{X}$ such that $x+0 = x$ for all $x \in \mathbb{X}$.
   d.) For each $x \in \mathbb{X}$ there is a unique element $-x \in \mathbb{X}$ such that $x+(-x)=0$.
2. For all $x, y \in \mathbb{X}$, $\lambda, \nu \in \mathbb{R}$
   a.) $1 \cdot x = x$
   b.) $(\lambda+\nu)x = \lambda x + \nu x$
   c.) $\lambda(\nu x) = (\lambda\nu)x$
   d.) $\lambda(x+y) = \lambda x + \lambda y$.

Definition - A norm on a linear space $\mathbb{X}$ is a function $\|\cdot\| : \mathbb{X} \mapsto \mathbb{R}$ satisfying

a.) $\|x\| \geq 0$
b.) $\|\lambda x\| = |\lambda| \cdot \|x\|$
c.) $\|x+y\| \leq \|x\| + \|y\|$
d.) $\|x\| = 0 \Leftrightarrow x = 0$.

* A normed linear space $(\mathbb{X}, \|\cdot\|)$ is a linear space with a norm $\|\cdot\|$. $(\mathbb{X}, \|\cdot\|)$ is a metric space with with

$$d(x,y) = \|x-y\|.$$

Definition - Let $1 \le p \le \infty$ and $n \in \mathbb{N}$. The ℓ^p norm on $\mathbb{R}^n$ is defined by

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \quad \text{for } 1 \le p < \infty$$

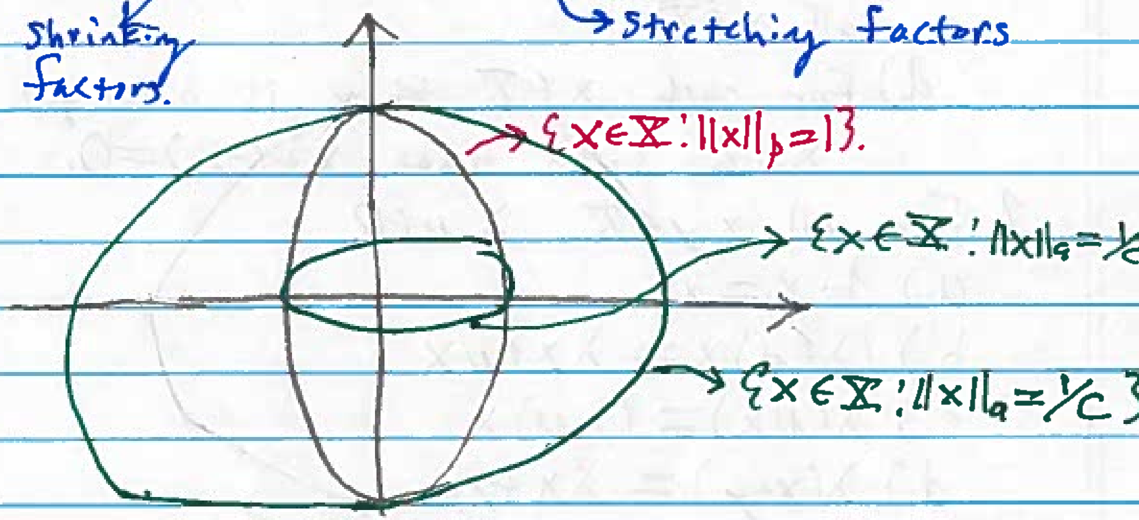
$$\|x\|_\infty = \max_{1 \le i \le n} \{|x_i|\}$$

Definition - We say two norms $\|\cdot\|_a$ and $\|\cdot\|_b$ on X are equivalent if for all $x \in X$:

$$c\|x\|_a \le \|x\|_b \le C\|x\|_a$$

Shrinking factors

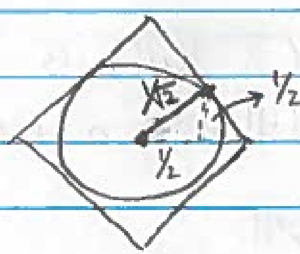
Stretching factors



Definition - Two metrics d and d' are equivalent if for every $x \in M$ and $\varepsilon > 0$, there exists δ and δ' such that

$$d(x,y) < \delta \Rightarrow d'(x,y) < \varepsilon$$
$$d'(x,y) < \delta' \Rightarrow d(x,y) < \varepsilon$$

example:



$$d_2(x,y) < 1/\sqrt{2} \Rightarrow d_1(x,y) < 1$$
$$\Rightarrow d_2(x,y) < \varepsilon/\sqrt{2} \Rightarrow d_1(x,y) < \varepsilon$$

$d_1(x,y)<1 \Rightarrow d_2(x,y)<1$

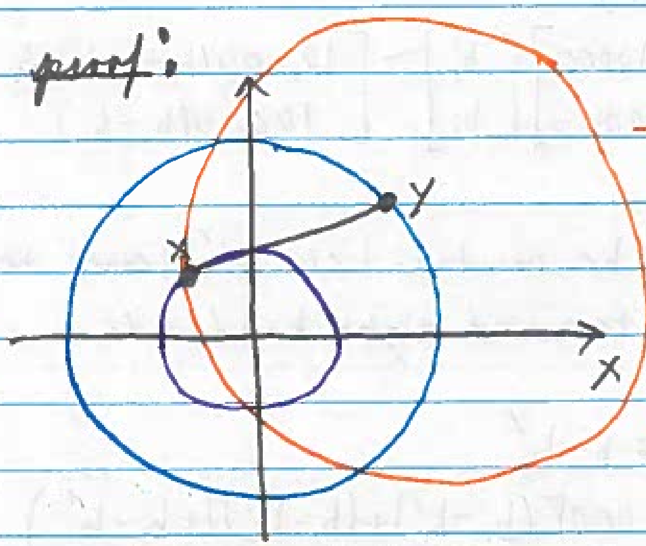
$\Rightarrow d_1(x,y)<\varepsilon \Rightarrow d_2(x,y)<\varepsilon$

Theorem - Let $(\mathbb{X}, \|\cdot\|)$ be a linear space with a mapping $\|\cdot\|$ satisfying

1. $\|x\| \geq 0$, $\|x\|=0 \Leftrightarrow x=0$.
2. $\|\lambda x\| = |\lambda| \cdot \|x\|$.

If $B_1(0)$ is convex then $(\mathbb{X}, \|\cdot\|)$ is a normed linear space.

proof:



$\rightarrow$ The orange ball's radius can be at most $\max\{2\|x\|, 2\|y\|\}$ i.e. twice the diameter of the blue ball and purple ball.

$\Rightarrow \|x-y\| \leq 2\cdot\|x\|$ and $\|x-y\| \leq 2\cdot\|y\|$

$\Rightarrow 2\|x-y\| \leq 2\cdot\|x\| + 2\|y\|$

$\Rightarrow \|x-y\| \leq \|x\| + \|y\|$ ∎

Theorem: All norms on $\mathbb{R}^n$ or equivalent.

proof:

Since all unit balls are convex and it is a finite dimensional space, they can be scaled to fit inside of each other.

(You will do a special case of this rigorously).

Application Norms of Matrices

Solve $Ax = b$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1.0001 \end{bmatrix}$$

$$\Rightarrow x = \begin{bmatrix} 10001 & -10000 \\ -10000 & 10000 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 10{,}000(b_1 - b_2) + b_1 \\ 10000(b_2 - b_1) \end{bmatrix}$$

However, b might be rounded. Let x' denote truncated solution b' denote truncated right hand side

$$Ax' = b'$$

$$\Rightarrow A(x - x') = b - b'$$

$$\Rightarrow x - x' = \begin{pmatrix} 10{,}000[(b_1 - b_1') + (b_2 - b_2')] + b_1 - b_1' \\ 10{,}000[(b_2 - b_2') - (b_1 - b_1')] \end{pmatrix}$$

Suppose $b_1 = b_1'$ (stored perfectly)

$$\Rightarrow x - x' = 10{,}000(b_2 - b_2')\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Suppose $b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $b' = \begin{pmatrix} 1 \\ 1.0001 \end{pmatrix}$

$$x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad x' = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow x - x' = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(error is large)!

Idea of Matrix Norm:

Whenever

$$b = Ax$$

$$\Rightarrow \|b\| \le \|A\| \cdot \|x\|$$

Not defined yet.

1. $A^{-1}(b-b') = (x-x')$

$$\Rightarrow \|x-x'\| \le \|A^{-1}\| \cdot \|b-b'\|$$

2. $Ax = b$

$$\Rightarrow \|b\| \le \|A\| \cdot \|x\|$$

$$\Rightarrow \frac{1}{\|x\|} \le \frac{\|A\|}{\|b\|}$$

3. $\frac{\|x-x'\|}{\|x\|} \le \underbrace{\|A^{-1}\| \cdot \|A\|}_{\text{Amplification factor of errors.}} \cdot \frac{\|b-b'\|}{\|b\|}$

$K(A) = \|A^{-1}\| \cdot \|A\|$ is called the condition number of A.

Definition - Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$. For a matrix A, define

$$\|A\| = \sup_{\|x\|=1} \|Ax\| =$$

example:

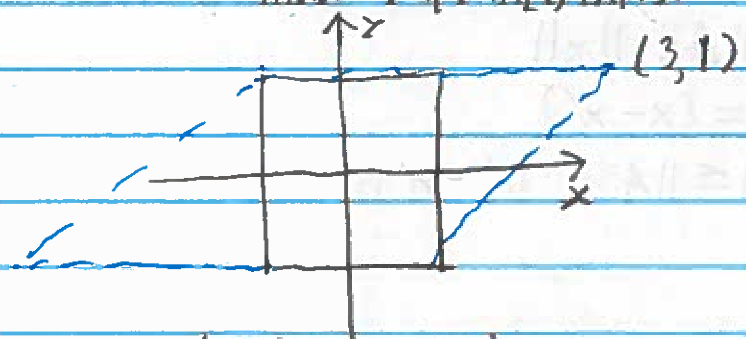
Start with ℓ^∞ norm on $\mathbb{R}^2$. By definition

$$\|A\|_\infty = \sup_{\|x\|_\infty=1} \|Ax\|_\infty.$$

$\|x\|_\infty = \max\{|x_1|, |x_2|\} = 1$

$\|Ax\|_\infty = \max\{|(Ax)_1|, |(Ax)_2|\}$

$= \max\{|x_1 + 2x_2|, |x_1|\}$



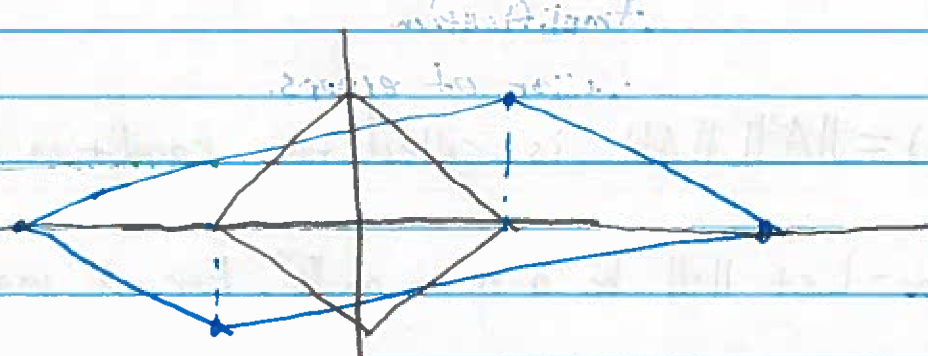
$\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$\Rightarrow \|A\|_\infty = 3.$

<u>example:</u>

Now compute

$\|A\|_1 = \max |x_1 + 2x_2| + |x_1|$ subject to $|x_1| + |x_2| = 1$



$\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

$\|A\|_1 = 2.$