Lecture 8: Ordinary Differential Equations

$$\frac{du}{dt} = f(t, u)$$

$f : \mathbb{R}^2 \to \mathbb{R}$ is continuous.

A solution is a continuously differentiable function $u : I \to \mathbb{R}$ such that

$$\frac{du}{dt} = f(t, u(t)),$$

where I is open interval.

$I \subset \mathbb{R}$ (local solution)

$I = \mathbb{R}$ (global solution).

Examples:

1. $\frac{du}{dt} = au \Rightarrow u(t) = u_0 e^{at}$.

$u(0) = u_0$

2. $\frac{du}{dt} = u^3 \Rightarrow \int_{u_0}^{u} \frac{1}{u^3}\,du = \int_0^t dt$

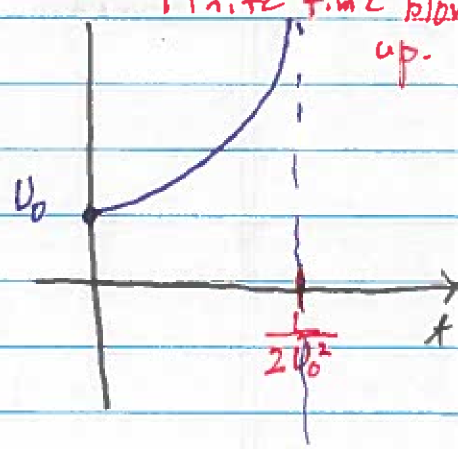
$u(0) = u_0 \quad \Rightarrow \frac{1}{2}\left(\frac{1}{u_0^2} - \frac{1}{u^2}\right) = t$

$\Rightarrow \frac{1}{u_0^2} - 2t = \frac{1}{u^2}$

$\Rightarrow \frac{1 - 2tu_0^2}{u_0^2} = \frac{1}{u^2}$

$\Rightarrow u(t) = \frac{u_0}{\sqrt{1 - 2tu_0^2}}$.

Finite time blow up.



Local Existence until $t = 1/2u_0^2$.

3. $\frac{du}{dt} = u^{1/3} \Rightarrow \int_0^u u^{-1/3}\,du = \int_0^t dt$

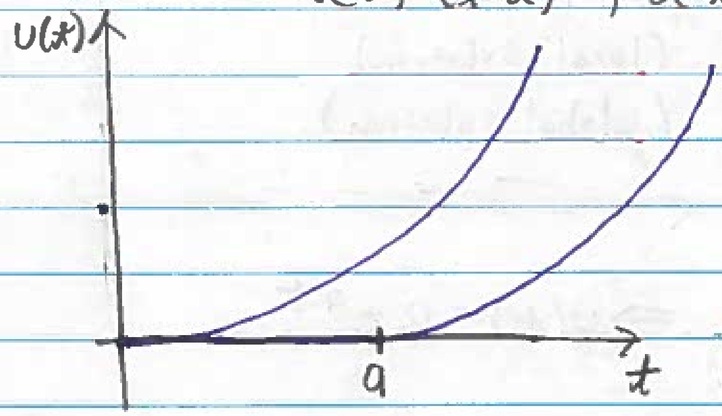
$u(0)=0 \quad \Rightarrow \frac{3}{2}(u^{2/3}) = t$

$\Rightarrow u^{2/3} = \frac{2}{3}t$

$\Rightarrow u(t) = \left(\frac{2}{3}\right)^{3/2} t^{3/2}.$

Not the only solution:

$$u(t) = \begin{cases} 0 & 0 \le t \le a \\ \left(\frac{2}{3}\right)^{3/2}(t-a)^{3/2}, & a < t < \infty \end{cases} \quad (a>0)$$



→ Solutions are not unique.

Theorem - Suppose $f(t,u)$ is a continuous function on $\mathbb{R}^2$. Then for every (t_0, u_0), there is an open interval $I \subset \mathbb{R}$ that contains t_0 and a continuously differentiable function $u: I \to \mathbb{R}$ that satisfies

$$\frac{du}{dt} = f(t,u)$$

$$u(t_0) = u_0.$$

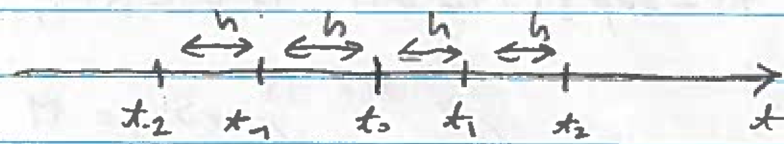
proof:

1. Pick $T_1 > 0$ and let

$$I_1 = \{t : |t - t_0| \le T_1\}$$

Partition I_1 into $2N$ subintervals of length h, where $T_1 = Nh$, and let

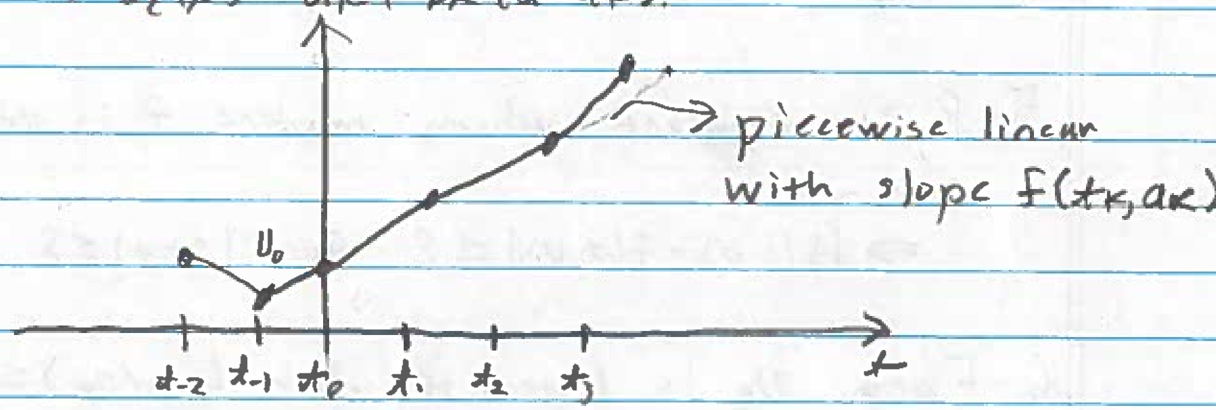
$$t_k = t_0 + kh \quad \text{for } -N \le k \le N$$



2. We construct an approximate solution $v_\varepsilon(t)$. Let $v_\varepsilon(t_k) = a_k$. Approximate

$$\left.\frac{dv_\varepsilon}{dt}\right|_{t_k} \simeq \frac{a_{k+1} - a_k}{h} = f(t_k, a_k).$$

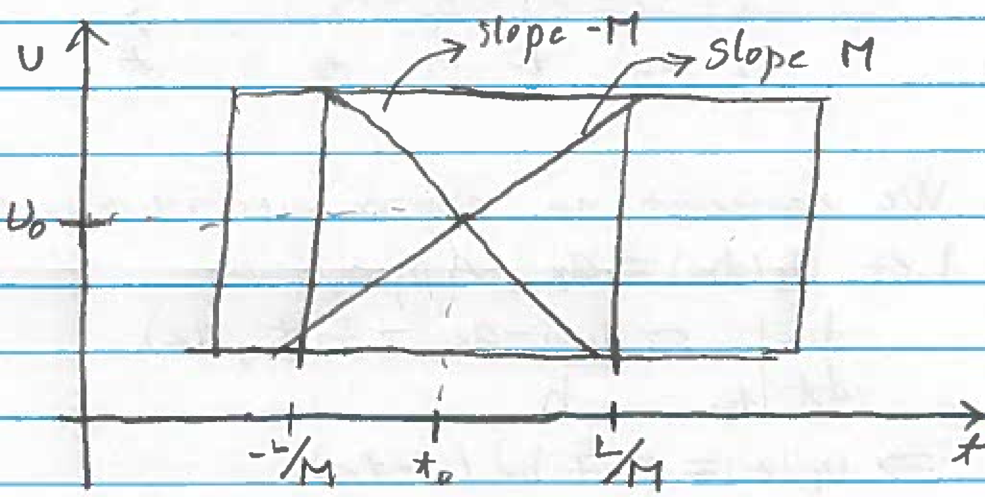
$$\Rightarrow v_\varepsilon(t) = a_k + b_k(t - t_k).$$



3. $\left|\frac{dv_\varepsilon}{dt} - f(t, v_\varepsilon(t))\right| = |f(t_k, a_k) - f(t, a_k + b_k(t - t_k))|$

$|t - t_k| \le h$, $\quad |a_k + b_k(t - t_k) - a_k| \le |b_k|\, h$

4. We choose an L and $T \leq T_1$ such that for all ε the graph of u_ε is contained in R given by
$$R = \{(t,u) : |t - t_0| \leq T,\ |u - u_0| \leq L\}.$$
Let R_1 be defined by:
$$R_1 = \{(t,u) : |t - t_0| \leq T_1,\ |u - u_0| \leq L\}.$$
Let $M = \sup\{|f(t,u)| : (t,u) \in R_1\}$, $T = \min\{T_1, L/M\}$



5. R is compact which implies f is uniformly continuous
$$\Rightarrow |f(s,u) - f(t,v)| \leq \varepsilon \quad \text{for } |s-t| \leq \delta \text{ and } |u-v| \leq \delta$$

6. Each u_ε is Lipschitz with $\mathrm{Lip}(u_\varepsilon) \leq M$.
By Arzela-Ascoli $u_{\varepsilon_k} \to u$.

7. $u_\varepsilon(t) = u_\varepsilon(t_0) + \int_{t_0}^{t} \frac{du_\varepsilon}{ds}\, ds$

$$= u_\varepsilon(t_0) + \int_{t_0}^{t} f(s, u_{\varepsilon_k}(s))\, ds + \int_{t_0}^{t} \left[\frac{du_{\varepsilon_k}}{ds} - f(s, u_{\varepsilon_k}(s))\right]$$
Take limit as $k \to \infty$.
$$u(t) = u(t_0) + \int_{t_0}^{t} f(s, u(s))\, ds.$$

Theorem - Suppose $u(t) \geq 0$, $\varphi(t) \geq 0$ are continuously differentiable functions defined on $0 \leq t \leq T$ and $u_0 > 0$. If

$$u(t) \leq u_0 + \int_0^t \varphi(s) u(s)\, ds.$$

then

$$u(t) \leq u_0 \exp\left(\int_0^t \varphi(s)\, ds\right)$$

proof:

Let $U = u_0 + \int_0^t \varphi(s) u(s)\, ds$. Then,

$$\frac{dU}{dt} = \varphi(t) \cdot u(t) \leq \varphi(t) \cdot U(t)$$

$$\Rightarrow \frac{d}{dt} \ln(U) = \frac{\frac{dU}{dt}}{U} \leq \varphi(t)$$

$$\Rightarrow \ln(U) = \ln(u_0) + \int_0^t \varphi(s)\, ds$$

$$\Rightarrow u(t) \leq U(t) \leq u_0 \exp\left(\int_0^t \varphi(s)\, ds\right).$$

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Theorem - If f is a Lipschitz continuous function of u uniformly in t then the solution of

$$* \qquad \frac{du}{dt} = f(t,u), \quad u(t_0) = u_0.$$

is unique.

proof:

Suppose u, v solve $*$. Therefore,

$$u(t) - v(t) = \int_{t_0}^t [f(s,u(s)) - f(s,v(s))]\, ds$$

$$\Rightarrow |u(t) - v(t)| \leq \int_{t_0}^t |f(s,u(s)) - f(s,v(s))|\, ds \leq C \int_{t_0}^t |u(s) - v(s)|$$

Therefore, by Gronwall's inequality,

$$|u(t) - v(t)| \leq |u(t_0) - v(t_0)| \exp\left(C \int_{t_0}^t |u(s) - v(s)|\, ds\right) = 0.$$