Lecture 9: Weierstrass Approximation Theorem

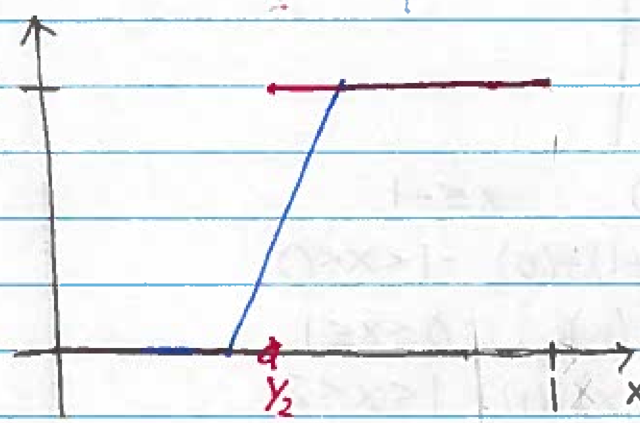
Average Value:

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

* Averaging can smooth out a function.

example:

$$f(x)=\begin{cases} 0, & 0\le x<\frac{1}{2} \\ 1, & \frac{1}{2}\le x\le 1 \end{cases}, \quad f_n(x)=\begin{cases} 0, & 0\le x<\frac{1}{2}-\frac{1}{n} \\ \frac{n}{2}\left(x-\frac{1}{2}+\frac{1}{n}\right), & \frac{1}{2}-\frac{1}{n}<x<\frac{1}{2}+\frac{1}{n} \\ 1, & \frac{1}{2}+\frac{1}{n}\le x\le 1 \end{cases}$$

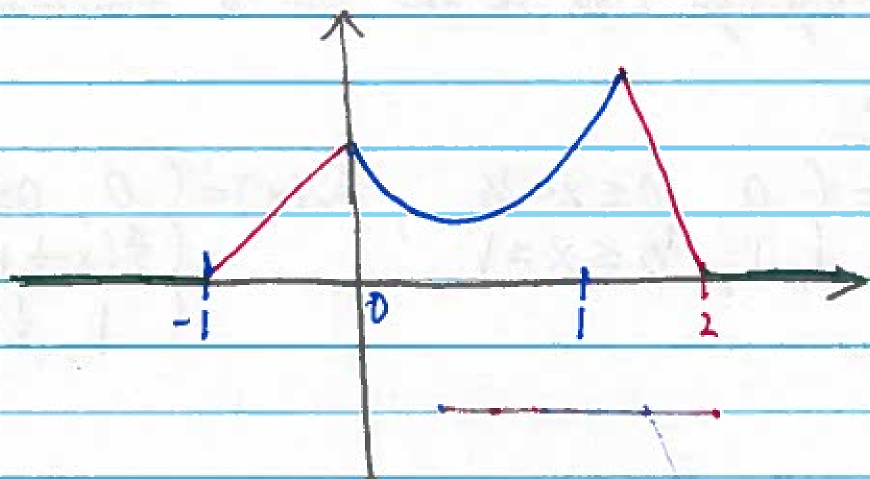


$$f_n(x)=\frac{1}{\frac{1}{n}+\frac{1}{n}}\int_{x-1/n}^{x+1/n} f(y)\,dy=\frac{n}{2}\int_{x+1/n}^{x-\frac{1}{n}} f(y)\,dy$$

$$\Rightarrow f_n(x)=\frac{n}{2}\begin{cases} 0 & 0\le x\le\frac{1}{2}-\frac{1}{n} \\ \int_{1/2}^{x+1/n} dy & \frac{1}{2}-\frac{1}{n}\le x\le\frac{1}{2}+\frac{1}{n} \\ \frac{2}{n}, & \frac{1}{2}+\frac{1}{n}<x\le 1 \end{cases}$$

$$=\begin{cases} 0 & 0\le x\le\frac{1}{2}-\frac{1}{n} \\ \frac{n}{2}\left(x-\frac{1}{2}+\frac{1}{n}\right), & \frac{1}{2}-\frac{1}{n}\le x\le\frac{1}{2}+\frac{1}{n} \\ 1 & , \frac{1}{2}+\frac{1}{n}<x\le 1. \end{cases}$$

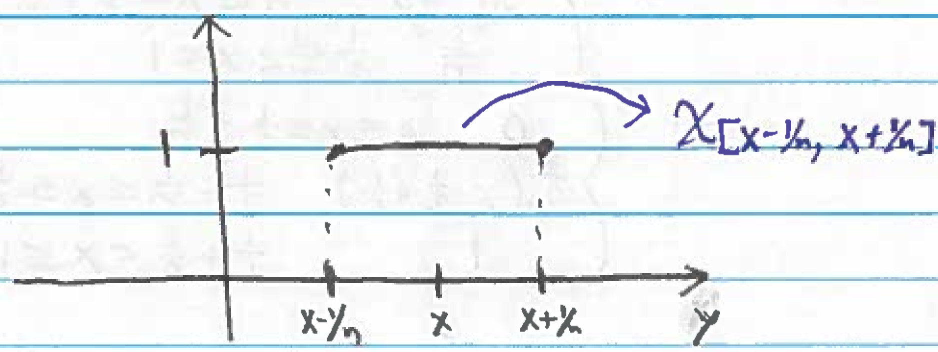
Problem - The Averaging integrates outside of the domain. Need to extend to a function with compact support.



$$\bar{f}(x) = \begin{cases} 0, & x \leq -1 \\ (x+1)f(0), & -1 < x < 0 \\ f(x), & 0 \leq x \leq 1 \\ (2-x)f(1), & 1 < x < 2 \\ 0, & x \geq 2 \end{cases}$$

Now,

$$\frac{n}{2}\int_{x-1/n}^{x+1/n} f(y)\,dy = \frac{n}{2}\int_{-\infty}^{\infty} \chi_{[x-1/n,\, x+1/n]} f(y)\,dy.$$



$$\chi_{[a,b]}(x) = \begin{cases} 0, & x \notin [a,b] \\ 1, & x \in [a,b] \end{cases}$$

$$\Rightarrow \frac{n}{2}\int_{x-1/n}^{x+1/n} \bar{f}(y)\,dy = \frac{n}{2}\int_{-\infty}^{\infty} \chi_{[x-1/n,\,x+1/n]}(y)\bar{f}(y)\,dy,$$

$$= \frac{n}{2}\int_{-\infty}^{\infty} \chi_{[-1/n,\,1/n]}(\underset{\uparrow}{y-x})\bar{f}(y)\,dy,$$

Horizontal Shift by x

$$= \int_{-\infty}^{\infty} g_n(y-x)\bar{f}(y)\,dy.$$

g_n ~ kernal of the approximation.

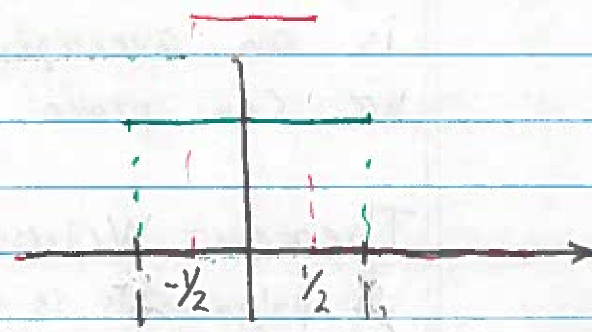
Approximation Process.

1. Extend f to $\bar{f}$ defined on all of $\mathbb{R}$.
2. Use appropriate kernel g_n
3. Convolve with g_n:

$$f_n(x) = \underbrace{(\bar{f} * g_n)}_{\text{Convolution}}(x) = \int_{-\infty}^{\infty} \bar{f}(y)g_n(x-y)\,dy,$$

Properties of g_n:

1. $g_n \geq 0$
2. $\int_{-\infty}^{\infty} g_n(y)\,dy = 1$
3. $\lim_{\delta\to 0}\left(\int_{[-\delta,\delta]^c} g(y)\,dy\right) = 0$



Theorem - Let g_n be a sequence of averaging kernals. Let $f_n = \bar{f} * g_n$, then f_n converges uniformly to $\bar{f}$.

~~proof~~:

$$|f_n(x) - \bar{f}(x)| = \left|\int_{-\infty}^{\infty} g_n(y)\bar{f}(x-y)\,dy - \bar{f}(x)\int_{-\infty}^{\infty} g_n(y)\,dy\right|$$

$$\leq \int_{-\infty}^{\infty} g_n(y)|\bar{f}(x-y) - \bar{f}(x)|\,dy$$

$$\leq \int_{[-\delta,\delta]} g_n(y)|\bar{f}(x-y) - \bar{f}(x)|\,dy + \int_{[-\delta,\delta]^c} g_n(y)|\bar{f}(x-y) - \bar{f}(x)|\,dy.$$

Unifrom continuity

$$\Rightarrow |f_n(x) - \bar{f}(x)| \leq \varepsilon \int_{[-\delta,\delta]} g_n(y)dy + \int_{[-\delta,\delta]^c} g_n(y)|\bar{f}(x-y)-\bar{f}(x)|dy$$

$$\leq \varepsilon + \int_{[-\delta,\delta]^c} g_n(y) \cdot 2\|\bar{f}\|_\infty dy$$

$$\Rightarrow \limsup_{n\to\infty} |f_n(x) - \bar{f}(x)| < \varepsilon.$$

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Define $p_n(x) = \begin{cases} (1-x^2/4)^n, & -2 \leq x \leq 2 \\ 0, & x \in [-2,2]^c \end{cases}$

and $c_n = \int_{-\infty}^{\infty} p_n(x)\,dx$, Therefore,

$$g_n(x) = \frac{1}{c_n} p_n(x)$$

is an averaging kernel. Using this kernel we can prove

Theorem - Weierstrass Approximation Theorem - The set of polynomials is dense in $C([0,1])$.