Quiz 7

Analysis

March 22, 2018

1. Consider the heat equation with periodic forcing:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \sin^2(t)u,$$
$$u(x,0) = u_0(x),$$
$$u(0,t) = u(L,t) = 0.$$

Assuming that this equation has a smooth solution $u(x,t)$ for all smooth initial conditions $u_0(x)$, prove that the "spatial" L^2 norm decays as a function of time and use this to prove the uniqueness of smooth solutions to this equation.

1. $\frac{d}{dt}\|u(x,t)\|_{L^2}^2 = 2\cdot\|u(x,t)\|_{L^2}\cdot\frac{d}{dt}\|u(x,t)\|_{L^2}$

$= \int_0^L 2u(x,t)\cdot u_t(x,t)\,dx$

$= \int_0^L 2u(x,t)\left(u_{xx}(x,t) - \sin^2(t)u(x,t)\right)dx$

$= -\int_0^L 2\left(u_x^2(x,t) + \sin^2(t)u^2(x,t)\right)dx$

$\leq 0.$

2. If we suppose $u(x,t)$, $v(x,t)$ are both solutions to this initial-boundary value problem then

$\|u(x,t) - v(x,t)\|_{L^2} = f(t)$

is monotonically decreasing in t. However $f(0)=0$ and thus $f(t)=0$ for all t.