Quiz 8

Analysis

April 6, 2018

This is a group assignment. You are encouraged to work together. You can use the board and talk to each other. However, you cannot use the Internet! Jessi is a fantastic resource. Feel free to ask her questions or ask for hints. The point of this assignment is to provide an introduction to discrete dynamical systems.

A discrete dynamical system on $D \subset \mathbb{R}$ is relationship between iterates $x_n \in D$ given by

$$x_{n+1} = f(x_n),$$

where $f : D \mapsto D$ is a function. Another commonly used terminology is an iterated map. Some standard definitions in this field are the following:

- The **orbit** $\gamma(x_0)$ of a point x_0 is a set defined by:
$$\gamma(x_0) = \{x_n : x_n = f(x_{n-1})\}.$$
- A **period** k **orbit** is a an orbit with cardinality k:
$$\gamma(x_0) = \{x_0, x_1, \ldots, x_{k-1}\}.$$
- x^* is a **fixed point** if $x^* = f(x^*)$.
- A fixed point x^* is **locally stable** if there exists an open interval $I \subset D$ containing x^* such that for all $x_0 \in I$, $\lim_{n\to\infty} f(x_n) = x^*$.
- A fixed point x^* is **globally stable** if for all $x_0 \in D$, $\lim_{n\to\infty} f(x_n) = x^*$.
- A fixed point x^* is **unstable** if it is not locally stable.

1. If x^* is a fixed point of f, what is $\gamma(x^*)$?

   $\gamma(x^*) = \{x^*\}$

2. If $\gamma(x_0)$ is a period 2 orbit of f, find a function g ~~for which x^* is a fixed point of.~~ such that x_0 is a fixed point of g.

   Let $g = f \circ f$; Then,
   $g(x_0) = f(f(x_0)) = f(x_1) = x_0$.

3. If x^* is a fixed point of f, prove that if $|f'(x^*)| < 1$, then x^* is locally stable. **Hint:** This is a contraction mapping type argument.

By the mean value theorem it follows that

$$|x_n - x^*| = |f(x_{n-1}) - f(x^*)| = |f'(c)|\,|x_{n-1} - x^*|$$

If $|f'(x^*)| < 1$, it follows from continuity that there exists I containing x^* such that for all $c \in I$, $|f'(c)| < 1$. Therefore on I, f is a contraction which proves stability.

4. Find and classify the fixed points as stable or unstable for the following iterated map on $\mathbb{R}$:

$$x_{n+1} = \frac{3}{2}x_n(1 - x_n).$$

Fixed points:

$x = 0$, $\quad 1 = \frac{3}{2}(1-x)$

$\Rightarrow x = \frac{1}{3}$

$f'(x) = \frac{3}{2} - 3x$

$f'(0) = \frac{3}{2}$, $f'(\frac{1}{3}) = \frac{1}{2}$

0 is unstable

$\frac{1}{3}$ is stable.

5. Come up with a definition for what it means for a period k orbit to be locally stable. **Hint:** Thinking of compositions of f might be useful.

A period K orbit $\gamma(x_0) = \{x_0, \ldots, x_{k-1}\}$ is stable if x_0 is a stable fixed point of $f^k(x)$.

6. Suppose $\gamma(x_0)$ is a period k orbit. Prove that $\gamma(x_0)$ is locally stable if

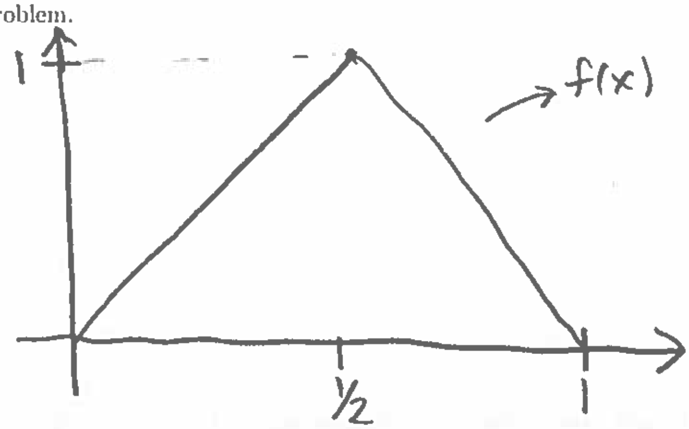
$$\prod_{i=0}^{k-1} |f'(x_i)| < 1.$$

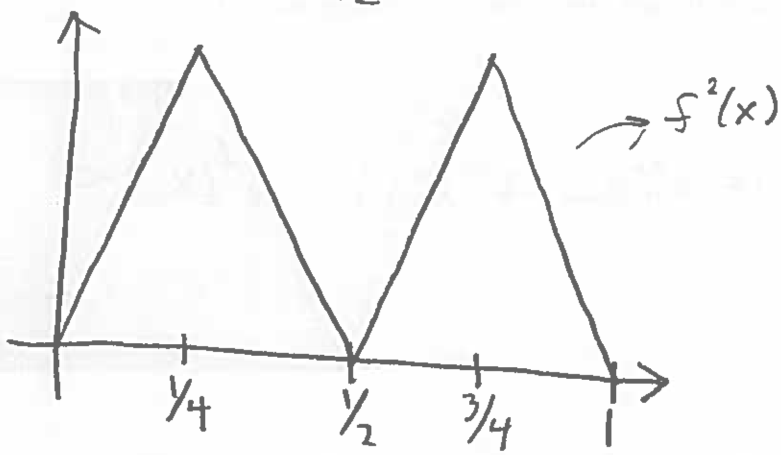
$$\frac{d}{dx} f^k(x) = f'(x_{k-1}) \cdot f'(x_{k-2}) \cdots f'(x_0) < 1.$$

7. Completely analyze the following iterated map on $[0, 1]$:

$$x_{n+1} = \begin{cases} 2x_n & \text{if } 0 \leq x_n \leq \frac{1}{2} \\ 2 - 2x_n & \text{if } \frac{1}{2} \leq x_n \leq 1 \end{cases},$$

i.e. determine the existence and stability of any fixed points or periodic orbits. Is there an attracting set for this problem? If so, what is it? **Hint:** Drawing f and its compositions could be very useful for this problem.





All fixed points and periodic orbits are unstable.
Attracting set is a topological Cantor set.