Homework #1: Solutions

#4.1

Show that if $f(p)$ has period p, the average value of f is the same over any interval of length p.

Solution:

Let a be any real number. It suffices to show that $\int_a^{a+p} f(x)\,dx = \int_0^p f(x)\,dx$. Calculating it follows that

$$\int_a^{a+p} f(x)\,dx = \int_a^p f(x)\,dx + \int_p^{a+p} f(x)\,dx$$

$$= \int_a^p f(x)\,dx + \int_0^a f(t+p)\,dt$$

$$= \int_a^p f(x)\,dx + \int_0^a f(t)\,dt$$

$$= \int_0^p f(x)\,dx.$$

#5.5

Find the Fourier series for

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ -1, & 0 < x < \pi/2 \\ 1, & \pi/2 < x < \pi \end{cases}$$

Solution:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi/2} (-1)\,dx + \frac{1}{\pi}\int_{\pi/2}^{\pi} (1)\,dx$$

$$= 0.$$

$$a_n = \frac{1}{\pi}\int_{\pi}^{\pi} \cos(nx) f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi/2} (-1)\cos(nx)\,dx + \frac{1}{\pi}\int_{\pi/2}^{\pi} \cos(nx)\,dx$$

$$= -\frac{1}{n\pi}\sin(nx)\Big|_0^{\pi/2} + \frac{1}{n\pi}\sin(nx)\Big|_{\pi/2}^{\pi}$$

$$= -\frac{2}{n\pi}\sin(n\pi/2)$$

$$= -\frac{2}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \dots\right)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(nx) f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi/2}(-1)\sin(nx)\,dx + \frac{1}{\pi}\int_{\pi/2}^{\pi}\sin(nx)\,dx$$

$$= -\frac{1}{n\pi}\cos(nx)\Big|_0^{\pi/2} - \frac{1}{n\pi}\cos(nx)\Big|_{\pi/2}^{\pi}$$

$$= \frac{1}{n\pi}\cos(n\pi/2) - \frac{1}{n\pi} - \frac{1}{n\pi}\cos(n\pi) + \frac{1}{n\pi}\cos\left(\frac{n\pi}{2}\right)$$

$$= \frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) - \frac{1}{n\pi}\left(1+(-1)^n\right)$$

$$= \frac{2}{\pi}\left(-\frac{1}{2}+\frac{1}{4}-\frac{1}{6}+\dots\right) - \frac{2}{\pi}\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\dots\right)$$

$$= -\frac{4}{\pi}\left(\frac{1}{2}+\frac{1}{6}+\frac{1}{10}+\dots\right)$$

Therefore,

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\cos((2n-1)x) - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n-2}\sin((4n-2)x)$$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\cos((2n-1)x) - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin((4n-2)x).$$

#7.13

If $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos(nx) + \sum_{n=1}^{\infty} b_n\sin(nx) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$, find a relationship between a_n, b_n, and c_n.

Solution:

$$\sum_{n=-\infty}^{\infty} c_n e^{inx} = \sum_{n=-\infty}^{-1} c_n e^{inx} + c_0 + \sum_{n=1}^{\infty} c_n e^{inx}$$

$$= \sum_{n=1}^{\infty} c_{-n}e^{-inx} + c_0 + \sum_{n=1}^{\infty} c_n e^{inx}$$

$$= c_0 + \sum_{n=1}^{\infty}\left[c_{-n}e^{-inx} + c_n e^{inx}\right]$$

$$= c_0 + \sum_{n=1}^{\infty}\left[c_{-n}(\cos(nx) - i\sin(nx)) + c_n(\cos(nx) + i\sin(nx))\right]$$

$$= c_0 + \sum_{n=1}^{\infty}(c_{-n}+c_n)\cos(nx) + \sum_{n=1}^{\infty} i(c_n - c_{-n})\sin(nx)$$

Therefore,

$$a_0 = c_0$$
$$a_n = (c_n + c_{-n})$$
$$b_n = i(c_n - c_{-n})$$

Inverting it follows that

$$c_0 = a_0$$
$$i a_n + b_n = 2i c_n$$
$$\Rightarrow c_n = \frac{a_n - i b_n}{2}$$
$$-i a_n + b_n = -2i c_{-n}$$
$$\Rightarrow c_{-n} = \frac{a_n + i b_n}{2}$$

#8.18

Find the Fourier series for $f(x) = x^2$ on the interval $0 < x < 10$.

Solution:

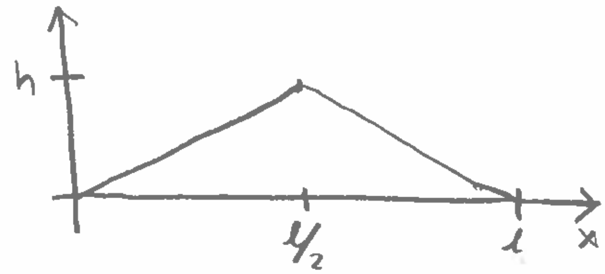
$$c_0 = \frac{1}{20}\int_0^{10} x^2\,dx$$
$$= \frac{1}{60} \cdot 10^3$$
$$= \frac{5}{3} \cdot 10$$
$$= \frac{50}{3}$$

$$c_n = \frac{1}{10}\int_0^{10} x^2 e^{\frac{in\pi x}{5}}\,dx$$
$$= \frac{1}{10}\left(\frac{x^2 5}{in\pi} e^{\frac{in\pi x}{5}}\Big|_0^{10} - \frac{5}{in\pi}\int_0^{10} 2x\, e^{\frac{in\pi x}{5}}\,dx\right.$$
$$= \frac{1}{10}\left(\frac{500}{in\pi} - \frac{10}{in\pi}\int_0^{10} x\, e^{in\pi x/5}\,dx\right)$$
$$= \frac{5}{in\pi} - \frac{1}{in\pi}\left(\frac{5}{in\pi} x e^{in\pi x/5}\Big|_0^{10} - \frac{5}{in\pi}\int_0^{10} e^{in\pi x/5}\,dx\right)$$
$$= \frac{5}{in\pi} + \frac{50}{n^2\pi^2}$$

$$\Rightarrow f(x) = \frac{50}{3} + \frac{5}{\pi}\sum_{n=-\infty}^{\infty}\left(\frac{1}{in\pi} + \frac{1}{n^2\pi^2}\right) e^{in\pi x/5}$$

#9.23

Find the Fourier sine series for the following function:



Solution:

Let $f(x)$ be defined by:

$$f(x) = \begin{cases} \frac{2h}{\ell}x, & 0 \le x \le \ell/2 \\ -\frac{2h}{\ell}x + 2h, & \ell/2 < x < \ell \end{cases}$$

Then,

$$b_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right)dx$$

$$= \frac{2}{\ell}\int_0^{\ell/2} \frac{2h}{\ell}x\sin\left(\frac{n\pi x}{\ell}\right)dx + \frac{2}{\ell}\int_{\ell/2}^{\ell}\left(-\frac{2h}{\ell}x + 2h\right)\sin\left(\frac{n\pi x}{\ell}\right)dx$$

$$= \frac{4h}{\ell^2}\int_0^{\ell/2} x\sin\left(\frac{n\pi x}{\ell}\right)dx - \frac{4h}{\ell^2}\int_{\ell/2}^{\ell} x\sin\left(\frac{n\pi x}{\ell}\right)dx + \frac{4h}{\ell}\int_{\ell/2}^{\ell}\sin\left(\frac{n\pi x}{\ell}\right)dx$$

$$= \frac{4h}{\ell^2}\left[\frac{\ell}{n\pi}x\left(-\cos\left(\frac{n\pi x}{\ell}\right)\right)\Big|_0^{\ell/2} + \frac{\ell}{n\pi}\int_0^{\ell/2}\cos\left(\frac{n\pi x}{\ell}\right)dx\right]$$
$$- \frac{4h}{\ell^2}\left[\frac{\ell}{n\pi}x\left(-\cos\left(\frac{n\pi x}{\ell}\right)\right)\Big|_{\ell/2}^{\ell} + \frac{\ell}{n\pi}\int_{\ell/2}^{\ell}\cos\left(\frac{n\pi x}{\ell}\right)dx\right]$$
$$- \frac{4h}{n\pi}\cos\left(\frac{n\pi x}{\ell}\right)\Big|_{\ell/2}^{\ell}$$

$$= \frac{4h}{\ell^2}\left[\frac{\ell^2}{2n\pi}\left(-\cos\left(\frac{n\pi}{2}\right)\right) + \frac{\ell^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right]$$
$$- \frac{4h}{\ell^2}\left[\frac{\ell^2}{n\pi}\left(-\cos(n\pi)\right) - \frac{\ell^2}{2n\pi}\left(-\cos\left(\frac{n\pi}{2}\right)\right) + \frac{\ell^2}{n^2\pi^2}\left[\sin(n\pi) - \sin\left(\frac{n\pi}{2}\right)\right]\right]$$
$$- \frac{4h}{n\pi}\left[\cos(n\pi) - \cos\left(\frac{n\pi}{2}\right)\right]$$

$$= -\frac{2h}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{4h}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) + \frac{4h}{n\pi}\cos(n\pi) - \frac{2h}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{4h}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$
$$- \frac{4h}{n\pi}\cos(n\pi) + \frac{4h}{n\pi}\cos\left(\frac{n\pi}{2}\right)$$

$$\Rightarrow b_n = \frac{8h}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right)$$

Therefore,

$$f(x) = \frac{8h}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \sin\left(\frac{(2n-1)\pi x}{\ell}\right).$$