Homework #4

pg. 459, #8.

$y'' + 4y' + 5y = \delta(t - t_0)$

$\Rightarrow (p^2 + 4p + 5)\mathcal{L}[y] = e^{-t_0 p}$

$\Rightarrow \mathcal{L}[y] = \dfrac{e^{-t_0 p}}{p^2 + 4p + 5}$

$\Rightarrow \mathcal{L}[y] = \dfrac{e^{-t_0 p}}{p^2 + 4p + 4 + 1}$

$\Rightarrow \mathcal{L}[y] = \dfrac{e^{-t_0 p}}{(p+2)^2 + 1}$

$$\Rightarrow y(t) = \delta(t - t_0) * e^{-2t}\sin(t)$$

$$= \int_0^t \delta(\tau - t_0) e^{-2(t-\tau)} \sin(t - \tau)\, d\tau$$

$$= \begin{cases} 0, & t < t_0 \\ e^{-2(t-t_0)}\sin(t - t_0), & t \geq t_0 \end{cases}$$

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$y'' - 9y = \delta(t - t_0)$

$\Rightarrow (p^2 - 9)\mathcal{L}[y] = e^{-t_0 p}$

$\Rightarrow \mathcal{L}[y] = \dfrac{e^{-t_0 p}}{(p-3)(p+3)}$

$$\Rightarrow y = \frac{\delta(t - t_0)}{3} * \sinh(3t)$$

$$= \int_0^t \frac{\delta(\tau - t_0)}{3} \sinh(3(t - \tau))\, d\tau$$

$$= \begin{cases} 0, & t \leq t_0 \\ \dfrac{\sinh(3(t - t_0))}{3}, & t > t_0 \end{cases}$$

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We prove this by induction. Assume

$$\int_{-\infty}^{\infty} \phi(x)\,\delta^{(n)}(x-a)\,dx = (-1)^n \phi^{(n)}(a).$$

Now,

$$\int_{-\infty}^{\infty} \phi(x)\,\delta^{(n+1)}(x-a)\,dx = \phi(x)\,\delta^{(n)}(x-a)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \phi'(x)\,\delta^{(n)}(x-a)\,dx$$

$$\Rightarrow \int_{-\infty}^{\infty} \phi(x)\,\delta^{(n+1)}(x-a)\,dx = (-1)\cdot(-1)^n \phi^{(n+1)}(a)$$

$$\Rightarrow \int_{-\infty}^{\infty} \phi(x)\,\delta^{(n+1)}(x-a)\,dx = (-1)^{n+1} \phi^{(n+1)}(a)$$

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$$\int_{-\infty}^{\infty} \phi(x)\,\frac{d}{dx}\operatorname{sgn}(x)\,dx = -\int_{-\infty}^{\infty} \phi'(x)\operatorname{sgn}(x)\,dx$$

$$= -\int_{-\infty}^{0} \phi'(x)(-1)\,dx - \int_{0}^{\infty} \phi'(x)\,1\,dx$$

$$= \phi(x)\Big|_{-\infty}^{0} - \phi(x)\Big|_{0}^{\infty}$$

$$= 2\phi(0)$$

Therefore, $\frac{d}{dx}\operatorname{sgn}(x) = 2\delta(x)$.

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a.) $\int_{0}^{3} (5x-2)\,\delta(2-x)\,dx = 8$

b.) $$\int_{0}^{\infty} \phi(x)\,\delta(x^2-a^2)\,dx = \int_{-a^2}^{\infty} \frac{\phi((u+a^2)^{1/2})\,\delta(u)\,du}{2x}$$

$$= \int_{-a^2}^{\infty} \frac{\phi((u+a^2)^{1/2})\,\delta(u)}{2(u+a^2)^{1/2}}\,du$$

$$= \frac{\phi(|a|)}{2|a|}$$

c.) $\int_{-1}^{1} \cos(x)\,\delta(-2x)\,dx = \frac{1}{2}.$

d.) $\int_{-\pi/2}^{\pi/2} \cos(x)\,\delta(\sin(x))\,dx = \int_{-1}^{1} \delta(u)\,du = 1.$