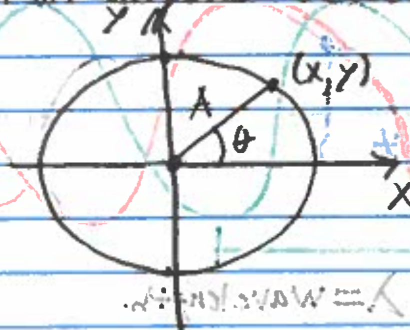
Lecture 1: Fourier Series I

Simple Harmonic Motion

P is a particle rotating on a circle of radius A at a constant angular velocity ω.



$$\theta = \omega t$$

$$\Rightarrow x = A\cos(\omega t), \quad y = A\sin(\omega t)$$

A – Amplitude of oscillation

$T = \frac{2\pi}{\omega}$ = Period of oscillation

We can also express in complex notation

$$z = x + iy = Ae^{i\omega t}$$

Example:

Determine the dynamics of a particle moving in the potential well

$$V(x) = Kx^2 \quad \text{(Hooke's Law)}$$

Newton's Law:

$$\frac{d^2x}{dt^2} = -V'(x) = -2Kx$$

$$x(0) = x_0$$

$$\left.\frac{dx}{dt}\right|_{t=0} = v_0$$

$$\Rightarrow x(t) = c_1\cos(\sqrt{2K}t) + c_2\sin(\sqrt{2K}t)$$

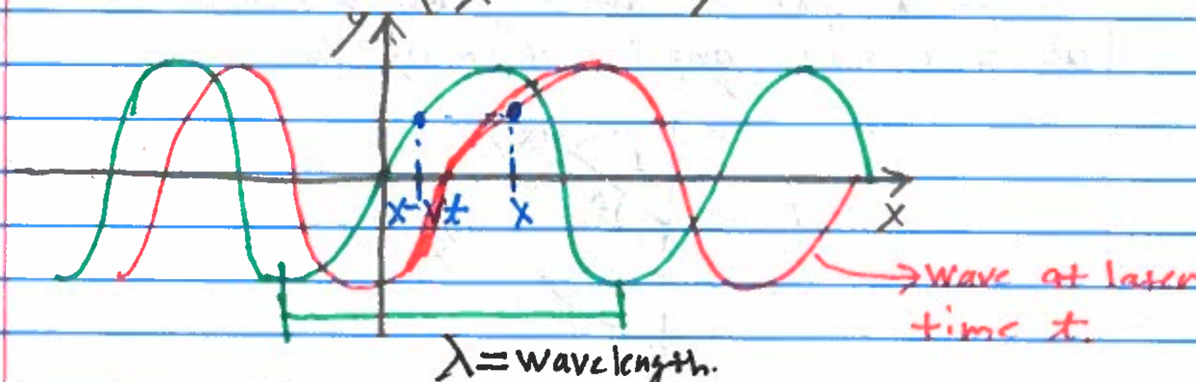
$$x(0) = 0 \Rightarrow c_1 = x_0$$

$$\left.\frac{dx}{dt}\right|_{t=0} = v_0 \Rightarrow c_2 = \frac{v_0}{\sqrt{2K}}$$

Example:

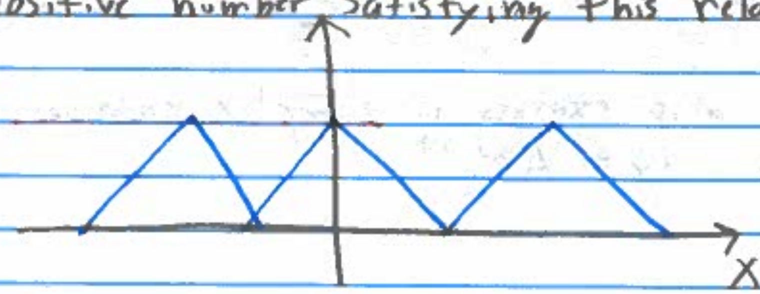
A travelling wave is given by

$$y = A\sin\left(\frac{2\pi}{\lambda}(x - vt)\right)$$



The temporal period is $T = \lambda/v$.

Definition - A function $f(x)$ is periodic of period p if for all x $f(x+p) = f(x)$, where p is the smallest positive number satisfying this relationship.



Average Value of a Function

Average of $f(x)$ on $[a,b]$

$$\approx \frac{f(x_1) + \dots + f(x_n)}{n}$$

$$= \frac{f(x_1) + \dots + f(x_n)}{n\Delta x}\Delta x$$

$$= \frac{1}{b-a}[f(x_1) + \dots + f(x_n)]\Delta x$$

$$\underset{\Delta x \to 0}{=} \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$\boxed{\text{Average of } f(x) \text{ on } [a,b] = \frac{1}{b-a}\int_a^b f(x)\,dx.}$$

Example:

The current in a wire is

$$I(t) = A\sin(\omega t)$$

The period is $T = 2\pi/\omega$. The root-mean-squared value of the current is given by:

$$R.M.S = \left(\frac{1}{2\pi/\omega}\int_0^{2\pi/\omega} A^2\sin^2(\omega t)\,dt\right)^{1/2}$$

$$= \sqrt{\frac{\omega}{2\pi}}\left(\int_0^{2\pi/\omega} A^2\sin^2(\omega t)\,dt\right)^{1/2}$$

$$= A\sqrt{\frac{\omega}{2\pi}}\left(\int_0^{2\pi/\omega}\left(\frac{1-\cos(2\omega t)}{2}\right)dt\right)^{1/2}$$

$$= A\sqrt{\frac{\omega}{2\pi}}\cdot\sqrt{\frac{\pi}{\omega}}$$

$$= \frac{A}{\sqrt{2\pi}}.$$

Fourier Coefficients

Assume $f(x)$ can be expanded in terms of trigonometric functions:

$$f(x) = \frac{1}{2}a_0 + a_1\cos(x) + a_2\cos(2x) + \ldots$$

$$+ b_1\sin(x) + b_2\sin(2x) + \ldots$$

How do we determine the coefficients??

1. $\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi}\left(\frac{1}{2}a_0 + \cancelto{0}{a_1\cos(x)} + \cancelto{0}{a_2\cos(2x)} + \ldots\right)dx$
$+ \int_{-\pi}^{\pi}\left(\cancelto{0}{b_1\sin(x)} + \cancelto{0}{b_2\cos(x)} + \ldots\right)dx$

(Integrals are zero from periodicity).

$$\Rightarrow \int_{-\pi}^{\pi} f(x)\,dx = \pi a_0$$

$$\Rightarrow a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

2. $\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx = \int_{-\pi}^{\pi}\left(\frac{1}{2}a_0\cos(mx) + \cos(x)\cos(mx) + \dots\right)dx$
$+\int_{-\pi}^{\pi}\left(b_1\cos(mx)\sin(x) + b_2\cos(mx)\sin(2x) + \dots\right)$

However,

$$\int_{-\pi}^{\pi}\cos(mx)\cos(nx)\,dx = \begin{cases} 0 & \text{if } m\neq n \text{ (periodicity)} \\ \int_{-\pi}^{\pi}\cos^2(nx) & \text{if } m=n \end{cases}$$

$$\Rightarrow \int_{-\pi}^{\pi}\cos(mx)\cos(nx)\,dx = \begin{cases} 0 & \text{if } m\neq n \\ \pi & \text{if } m=n \end{cases}$$

Therefore,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

3. Similarly,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx.$$

Example:

$$f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}$$

$$\Rightarrow a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0$$

$$\Rightarrow a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0}(-1)\cos(nx)\,dx + \frac{1}{\pi}\int_{0}^{\pi}\cos(nx)\,dx$$
$$= \frac{1}{n\pi}\left[\sin(-n\pi) + \sin(n\pi)\right]$$
$$= 0$$

$$\Rightarrow b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0}(-1)\sin(nx)\,dx + \frac{1}{\pi}\int_{0}^{\pi}\sin(nx)\,dx$$
$$= \frac{1}{n\pi}\left[\cos(nx)\Big|_{-\pi}^{0} - \cos(nx)\Big|_{0}^{\pi}\right]$$

$$= \frac{1}{n\pi}\left[1 - \cos(-n\pi) - \cos(n\pi) + 1\right]$$
$$= \frac{2}{n\pi}\left[1 - (-1)^n\right]$$

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[1-(-1)^n\right]\sin(nx)$$

$$= \frac{4}{\pi}\left(\sin(x) + \frac{\sin(3x)}{3} + \frac{\sin(5x)}{5} + \dots\right)$$