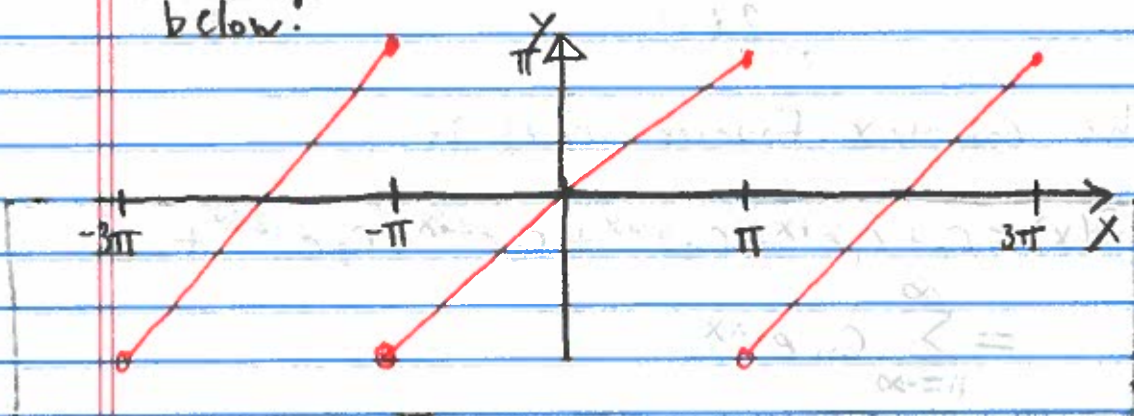
Lecture 2: Fourier Series II

Dirichlet Conditions

Example:

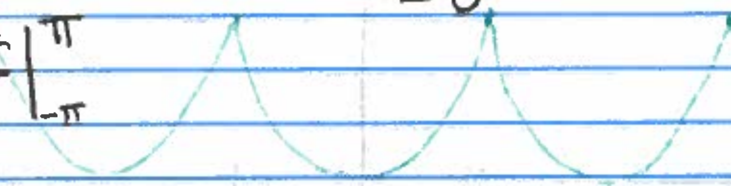
Find the Fourier series for the function graphed below:



$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \quad , \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{x\cos(nx)}_{\text{odd function}}\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx \qquad\qquad = 0$$

$$= \frac{1}{\pi}\,\frac{x^2}{2}\Big|_{-\pi}^{\pi}$$



$$= 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{x\sin(nx)}_{\text{even function}}\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x\sin(nx)\,dx$$

$$= \frac{2}{\pi}\left(\frac{-x\cos(nx)}{n}\right)\Bigg|_0^{\pi} + \frac{2}{n\pi}\underbrace{\int_0^{\pi}\cos(nx)\,dx}_{0 \text{ from periodicity.}}$$

$$= -\frac{2}{n}\cos(n\pi)$$

$$= \frac{2}{n}(-1)^{n+1}$$

The Fourier series is

$$\boxed{f(x) = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1}\sin(nx)}$$

Theorem - The Fourier series converges to the midpoint of a jump discontinuity.

Complex Fourier Series

Recall

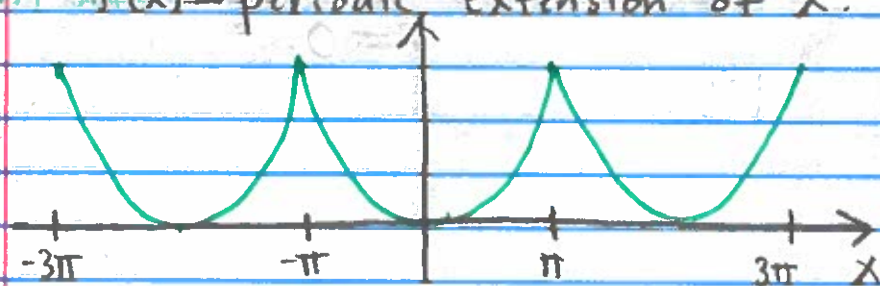
$$\sin(nx) = \frac{e^{inx} - e^{-inx}}{2i}, \quad \cos(nx) = \frac{e^{inx} + e^{-inx}}{2}$$

The complex Fourier series is

$$\boxed{\begin{aligned} f(x) &= c_0 + c_1 e^{ix} + c_{-1} e^{-ix} + c_2 e^{2ix} + c_3 e^{-2ix} + \dots \\ &= \sum_{n=-\infty}^{\infty} c_n e^{inx} \end{aligned}}$$

Example:

$f(x)$ = periodic extension of x^2.



How to find c_n?

$$x^2 = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$\Rightarrow \int_{-\pi}^{\pi} e^{-imx} x^2 \, dx = \sum_{n=-\infty}^{\infty} c_n \int_{-\pi}^{\pi} e^{-imx} e^{inx} \, dx$$

$$\int_{-\pi}^{\pi} e^{-imx} e^{inx} \, dx = \begin{cases} \int_{-\pi}^{\pi} dx, & m = n \\ \int_{-\pi}^{\pi} e^{i(n-m)x} \, dx & m \neq n \end{cases}$$

$$\Rightarrow \int_{-\pi}^{\pi} e^{-imx} e^{inx}\,dx = \begin{cases} 2\pi, & m=n \\ 0, & m\neq n \end{cases}$$

Therefore,

$$\int_{-\pi}^{\pi} x^2 e^{-imx}\,dx = 2\pi c_m$$

$$\Rightarrow 2\pi c_m = -\frac{x^2}{im} e^{-imx}\Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{2x}{im} e^{-imx}\,dx$$

$$= -\frac{1}{m}\cancelto{0}{\left(\pi^2 e^{-im\pi} - \pi^2 e^{im\pi}\right)} - \frac{2x}{(im)^2} e^{-imx}\Big|_{-\pi}^{\pi} + \cancelto{0}{\int_{-\pi}^{\pi} \frac{2}{(im)^2} e^{-imx}\,dx}$$

$$= \frac{2}{m^2}\left(\pi e^{-im\pi} + \pi e^{im\pi}\right)$$

$$= \frac{4\pi}{m^2}(-1)^m$$

$$\Rightarrow c_m = \frac{2(-1)^m}{m^2}, \quad c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{\pi^2}{3}$$

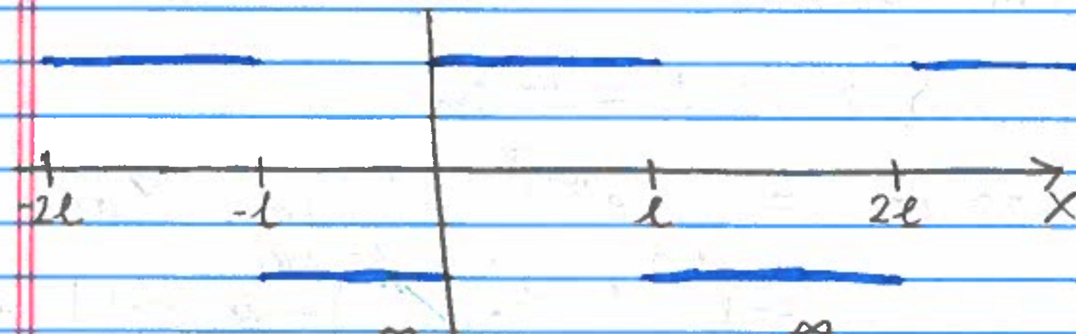
$$\Rightarrow f(x) = \sum_{n=-\infty}^{-1} \frac{2(-1)^n}{n^2} e^{inx} + \sum_{n=1}^{\infty} \frac{2(-1)^n e^{inx}}{n^2} + \frac{\pi^2}{3}$$

$$= \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^2} e^{-inx} + \sum_{n=1}^{\infty} \frac{2(-1)^n e^{inx}}{n^2} + \frac{\pi^2}{3}$$

$$= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos(nx)$$

Other Intervals:

$$f(x) = \begin{cases} 1 & 0 < x < \ell \\ -1 & \ell < x < 2\ell \end{cases}$$



$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\ell}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right)$$

Odd function $\Rightarrow a_0 = a_n = 0$

$$\Longrightarrow \int_{-\ell}^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right)dx = \int_{-\ell}^{\ell} b_n \sin^2\left(\frac{n\pi x}{\ell}\right)dx$$

$$\Longrightarrow 2\int_0^{\ell} \sin\left(\frac{n\pi x}{\ell}\right)dx = b_n \int_{-\ell}^{\ell} b_n \left(\frac{1-\cos\left(\frac{2n\pi x}{\ell}\right)}{2}\right)dx$$

$$\Longrightarrow -\frac{2\ell}{n\pi}\cos\left(\frac{n\pi x}{\ell}\right)\Big|_0^{\ell} = b_n \ell$$

$$\Rightarrow -\frac{2}{n\pi}\left((-1)^n - 1\right) = b_n$$

$$\Rightarrow b_n = \frac{2}{n\pi}\left(1 + (-1)^{n+1}\right)$$