Lecture 3: Parseval's Theorem and Fourier Transform.

Parseval's Theorem

$f(x)$ is periodic with period π

$$\Rightarrow f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x)^2 dx = \int_{-\pi}^{\pi} \left(\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)\right)^2 dx$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x)^2 dx = \int_{-\pi}^{\pi} \frac{1}{4}a_0^2 dx + \sum_{n=1}^{\infty}\int_{-\pi}^{\pi} a_n^2 \cos^2(nx)\, dx + \sum_{n=1}^{\infty}\int_{-\pi}^{\pi} b_n^2 \sin^2(nx)\, dx$$

$$= \frac{\pi}{2} a_0^2 + \pi \sum_{n=1}^{\infty} a_n^2 + \pi \sum_{n=1}^{\infty} b_n^2$$

$$\Rightarrow \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)^2 dx}_{\text{Average Energy}} = \frac{1}{4}a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2 + \frac{1}{2}\sum_{n=1}^{\infty} b_n^2$$

Energy in each mode.

Fourier Transform

Definition -

$$g(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{-i\alpha x} dx \qquad \text{(Fourier Transform)}$$

$$f(x) = \int_{-\infty}^{\infty} g(\alpha) e^{i\alpha x} d\alpha \qquad \text{(Inverse Fourier Transform)}$$

↓

Continuum sum of waves.

Other Notation:

$$\hat{f}(\alpha) = \mathcal{F}[f](\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{-i\alpha x} dx$$

$$f(x) = \mathcal{F}^{-1}[\hat{f}](x) = \int_{-\infty}^{\infty} \hat{f}(\alpha) e^{i\alpha x} d\alpha.$$

Example:

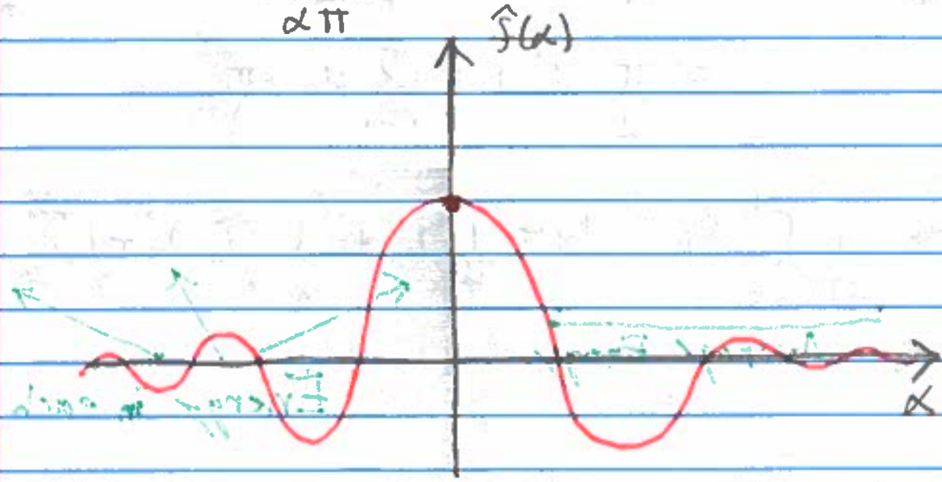
1. $f(x) = \begin{cases} 1, & -\pi < x < \pi \\ 0, & \text{o.w.} \end{cases}$

$$\mathcal{F}[f] = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{-i\alpha x}\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i\alpha x}\,dx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} \cos(\alpha x)\,dx$$

$$= \frac{\sin(\alpha\pi)}{\alpha\pi}$$



2. $f(x) = e^{-|x|}$

$$\mathcal{F}[f] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{|x|} e^{i\alpha x}\,dx$$

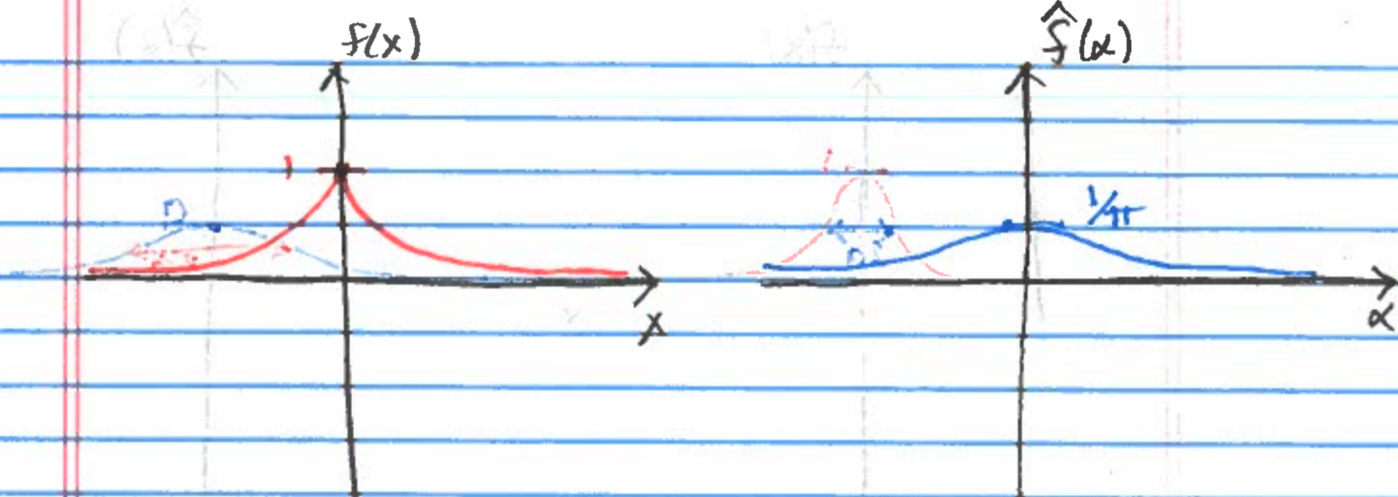
$$= \frac{1}{2\pi}\int_{0}^{\infty} e^{-x} e^{-i\alpha x}\,dx + \frac{1}{2\pi}\int_{-\infty}^{0} e^{x} e^{-i\alpha x}\,dx$$

$$= \frac{1}{2\pi}\int_{0}^{\infty} e^{-(1+i\alpha)x}\,dx + \frac{1}{2\pi}\int_{-\infty}^{0} e^{(1-i\alpha)x}\,dx$$

$$= \frac{-1}{2\pi(1+i\alpha)} e^{-(1+i\alpha)x}\Big|_{0}^{\infty} + \frac{1}{2\pi(1-i\alpha)} e^{(1-i\alpha)x}\Big|_{-\infty}^{0}$$

$$= \frac{1}{2\pi(1+i\alpha)} + \frac{1}{2\pi(1-i\alpha)}$$

$$= \frac{(1-i\alpha)+(1+i\alpha)}{2\pi(1+\alpha^2)} = \frac{1}{\pi(1+\alpha^2)}$$



3. $f(x) = e^{-x^2/a^2}$

$$\mathcal{F}[f] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-x^2/a^2} e^{-i\alpha x}\,dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left(\frac{-1}{a^2}\left(x^2 + i\alpha a^2 x\right)\right)dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left(\frac{-1}{a^2}\left(x^2 + i\alpha a^2 x + \frac{i^2\alpha^2 a^4}{4}\right)\right)\exp\left(\frac{i^2\alpha^2 a^2}{4}\right)dx$$

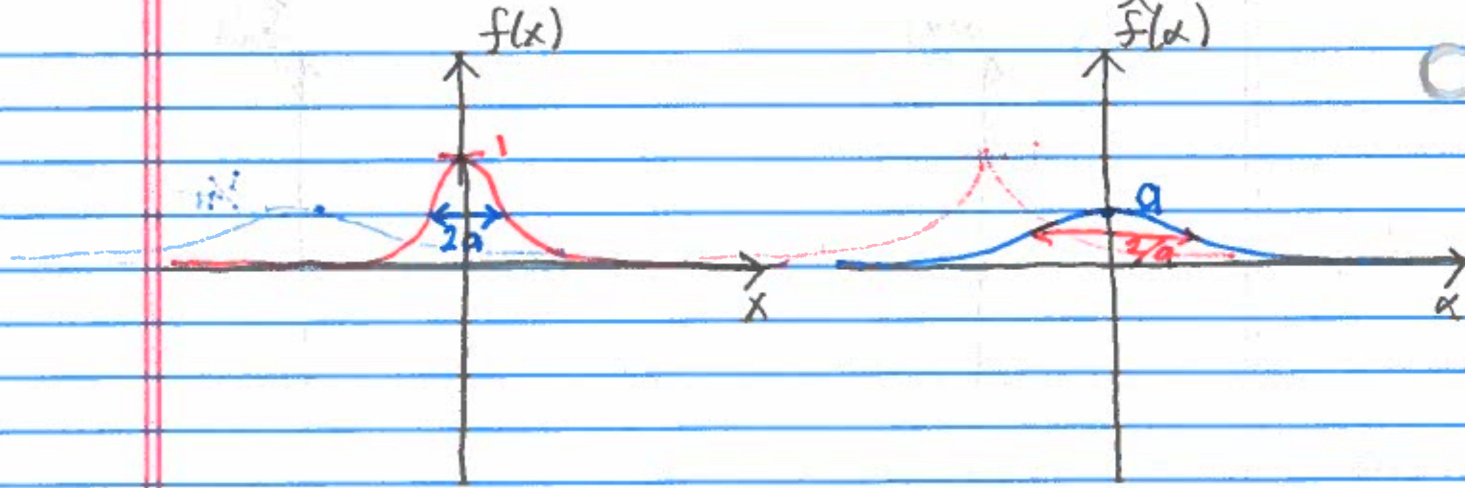
$$= \frac{1}{2\pi}\exp\left(-\frac{\alpha^2 a^2}{4}\right)\int_{-\infty}^{\infty}\exp\left(-\frac{1}{a^2}\left(x - \frac{i\alpha a^2}{2}\right)\right)dx$$

Let $u = \frac{1}{a}\left(x - \frac{i\alpha a^2}{2}\right) \Rightarrow du = \frac{1}{a}dx$

$$\Rightarrow \mathcal{F}[f] = \frac{a}{2\pi}\exp\left(-\frac{a^2}{4}\alpha^2\right)\int_{-\infty}^{\infty}\exp(-u^2)\,du$$

$$= \frac{a\sqrt{\pi}}{2\pi}\exp\left(\frac{-a^2}{4}\alpha^2\right)$$

$$\Rightarrow \mathcal{F}[f] = \frac{a}{2\sqrt{\pi}}\exp\left(\frac{-a^2}{4}\alpha^2\right)$$



Properties:

1. $$\mathcal{F}^{-1}[f(\alpha)](x) = \int_{-\infty}^{\infty} f(\alpha) e^{i\alpha x}\, d\alpha$$
$$= 2\pi \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} f(-\alpha) e^{-i\alpha x}\, d\alpha$$
$$= 2\pi\, \mathcal{F}[f(-\alpha)](x)$$

Example:

$$\mathcal{F}^{-1}[e^{-\alpha^2}](x) = 2\pi\, \mathcal{F}[e^{-\alpha^2}](x)$$
$$= 2\pi \cdot \frac{1}{2\sqrt{\pi}} e^{-x^2/4}$$
$$= \sqrt{\pi}\, e^{-x^2/4}.$$

2. $$\mathcal{F}[f(x-a)](\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x-a) e^{i\alpha x}\, dx$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u) e^{i\alpha(u+a)}\, du$$
$$= e^{i\alpha a}\, \mathcal{F}[f](\alpha).$$