Lecture 4: The Laplace Transform

$$\mathcal{L}[f](p) = \int_0^\infty f(t)e^{-pt}\,dt.$$

Examples:

1. $f(t) = 1$

$$\Rightarrow \mathcal{L}[f](p) = \int_0^\infty e^{-pt}\,dt$$
$$= -\frac{1}{p}e^{-pt}\Big|_0^\infty$$
$$= \frac{1}{p}.$$

2. $f(t) = \sin(t)$

$$\Rightarrow \mathcal{L}[f](p) = \int_0^\infty e^{-pt}\sin(t)\,dt$$
$$= \int_0^\infty e^{-pt}\left(\frac{e^{it}-e^{-it}}{2i}\right)dt$$
$$= \frac{1}{2i}\int_0^\infty \left[e^{-(p-i)t} - e^{-(p+i)t}\right]dt$$
$$= \frac{1}{2i}\left(\frac{1}{-(p-i)}e^{-(p-i)t} - \frac{1}{-(p+i)}e^{-(p+i)t}\right)\Big|_0^\infty$$
$$= \frac{1}{2i}\left(\frac{1}{(p-i)} - \frac{1}{(p+i)}\right)$$
$$= \frac{1}{2i}\left(\frac{p+i-p+i}{p^2+1}\right)$$
$$= \frac{1}{1+p^2}$$

Properties:

1. $\mathcal{L}[cf] = \int_0^\infty e^{-pt}cf(t)\,dt = c\int_0^\infty e^{-pt}f(t)\,dt = c\mathcal{L}[f]$
2. $\mathcal{L}[f+g] = \int_0^\infty e^{-pt}(f(t)+g(t))\,dt = \int_0^\infty e^{-pt}f(t)\,dt + \int_0^\infty e^{-pt}g(t)\,dt$

$$\Rightarrow \mathcal{L}[f+g] = \mathcal{L}[f] + \mathcal{L}[g].$$

3. $\mathcal{L}[f'] = \int_0^\infty e^{-pt}f'(t)\,dt$

$$= e^{-pt}f(t)\Big|_0^\infty + p\int_0^\infty e^{-pt}f(t)\,dt$$
$$= -f(0) + p\mathcal{L}[f].$$

4. $\mathcal{L}[f(at)] = \int_0^\infty e^{-pt} f(at)\,dt$

let $u = at$

$\Rightarrow du = a\,dt$

$$\Rightarrow \mathcal{L}[f(at)] = \frac{1}{a}\int_0^\infty e^{-p/a\,u} f(u)\,du$$

$$= \frac{1}{a}\mathcal{L}[f](p/a).$$

Examples

1. Given $\mathcal{L}\left[\frac{\sin(t)}{t}\right] = \tan^{-1}\left(\frac{1}{p}\right)$, find

$$\mathcal{L}\left[\frac{b\sin(at)}{t}\right] = \mathcal{L}\left[ab\,f(at)\right],$$

where $f(t) = \frac{\sin(t)}{t}$. Therefore,

$$\mathcal{L}[ab\,f(at)] = ab\,\mathcal{L}[f(at)]$$
$$= b\,\mathcal{L}[f](p/a)$$

$$= b\tan^{-1}\left(\frac{a}{p}\right)$$

2. Find $\mathcal{L}^{-1}\left[\frac{1}{p^3-p}\right] = \mathcal{L}^{-1}\left[\frac{1}{p(p^2-1)}\right]$

$$= \mathcal{L}^{-1}\left[\frac{1}{p(p-1)(p+1)}\right]$$

Now, $\frac{1}{p(p-1)(p+1)} = \frac{A}{p} + \frac{B}{p-1} + \frac{C}{p+1}$

$$= \frac{A(p-1)(p+1) + Bp(p+1) + Cp(p-1)}{p(p-1)(p+1)}$$

$\Rightarrow A = -1$

$B = 1/2$

$C = 1/2$

$$\Rightarrow \mathcal{L}^{-1}\left[\frac{1}{p^3-p}\right] = \mathcal{L}^{-1}\left[-\frac{1}{p} + \frac{1}{2(p-1)} + \frac{1}{2(p+1)}\right] = -1 + \frac{1}{2}e^{t} + \frac{1}{2}e^{-t}.$$

Solving Differential Equations

Examples

1. $y'' + 4y' + 4y = e^{-2t}$
   $y(0) = 0$
   $y'(0) = 4$

$$\mathcal{L}[y'' + 4y' + 4y] = \mathcal{L}[e^{-2t}]$$

$$\Rightarrow \mathcal{L}[y''] + 4\mathcal{L}[y'] + 4\mathcal{L}[y] = \frac{1}{p+2}$$

$$\Rightarrow p^2\mathcal{L}[y] - py(0) - y'(0) + 4p\mathcal{L}[y] - y(0) + 4\mathcal{L}[y] = \frac{1}{p+2}$$

$$\Rightarrow p^2\mathcal{L}[y] - 4 + 4p\mathcal{L}[y] + 4\mathcal{L}[y] = \frac{1}{p+2}$$

$$\Rightarrow (p^2 + 4p + 4)\mathcal{L}[y] = \frac{1}{p+2} + 4$$

$$\Rightarrow \mathcal{L}[y] = \frac{1}{(p+2)^3} + \frac{4}{(p+2)^2}$$

$$= \frac{1}{2}\frac{2}{(p+2)^3} + 4 \cdot \frac{1}{(p+2)^2}$$

$$\Rightarrow y = \frac{1}{2}t^2 e^{-2t} + 4te^{-2t},$$

where we have used the fact that

$$\mathcal{L}[t^k e^{-at}] = \frac{k!}{(p+a)^{k+1}}$$

Convolutions:

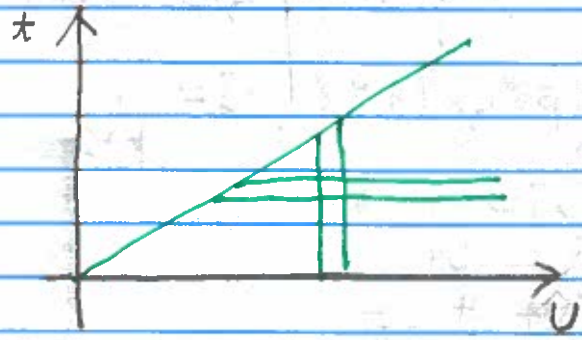
$$\mathcal{L}[f]\cdot\mathcal{L}[g] = \int_0^\infty e^{-pt} g(t)\,dt \cdot \int_0^\infty e^{-pt} h(t)\,dt$$

$$= \int_0^\infty e^{-p\tau} g(\tau)\,d\tau \int_0^\infty e^{-pt} h(t)\,dt$$

$$= \int_0^\infty \int_0^\infty e^{-p(\tau+t)} g(\tau) h(t)\,d\tau\,dt$$

Let $u = \tau + t$

$$\Rightarrow \mathcal{L}[f]\cdot\mathcal{L}[g] = \int_0^\infty \int_t^\infty e^{-pu} g(u-t) h(t)\,du\,dt$$



$$\Rightarrow \mathcal{L}[f]\cdot\mathcal{L}[g] = \int_0^\infty \int_0^t e^{-pu} g(u-t) h(t)\,dt\,du$$

$$= \int_0^\infty e^{-pu}\left[\int_0^t g(u-t) h(t)\,dt\right] du$$

$$= \mathcal{L}\left[\int_0^t g(u-t)\,h(t)\,dt\right]$$

Definition

The convolution of two functions f, g is defined by

$$f * g = \int_0^t f(t-\tau)\,g(\tau)\,d\tau$$

Theorem -

1. $\mathcal{L}[f*g] = \mathcal{L}[f]\cdot\mathcal{L}[g]$
2. $\mathcal{L}^{-1}[\mathcal{L}[f]\cdot\mathcal{L}[g]] = f*g.$

Example:

$$y'' + 3y' + 2y = e^{-t}$$
$$y(0) = 0$$
$$y'(0) = 0$$

$$\Rightarrow p^2 \mathcal{L}[y] + 3p\mathcal{L}[y] + 2\mathcal{L}[y] = \mathcal{L}[e^{-t}]$$

$$\Rightarrow \mathcal{L}[y] = \frac{1}{p^2+3p+2} \mathcal{L}[e^{-t}]$$

$$= \frac{1}{(p+2)(p+1)} \mathcal{L}[e^{-t}]$$

$$= \mathcal{L}[e^{-t} - e^{-2t}] \mathcal{L}[e^{-t}]$$

$$\Rightarrow y(t) = \int_0^t (e^{-\tau} - e^{-2\tau}) e^{-(t-\tau)} d\tau$$

$$= te^{-t} + 2e^{-2t} - e^{-t}.$$