Lecture 5: The Dirac Delta Function

Motivating Example:

$$* \quad \frac{d^2y}{dt^2} + y = f_n(t), \quad f_n(t) = \begin{cases} n, & 0 \le t \le 1/n \\ 0, & \text{o.w.} \end{cases}$$

$$y(0) = 0$$

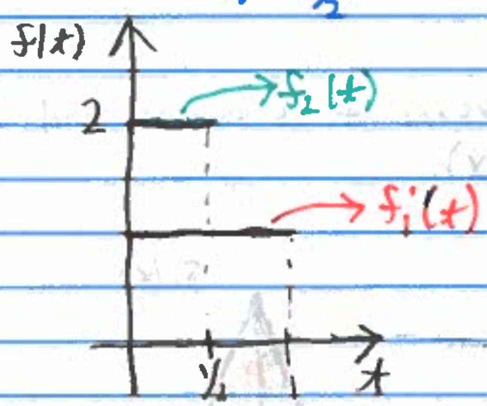
$$\left.\frac{dy}{dt}\right|_0 = 0$$

Newtons Law:

$$\frac{d^2y}{dt} = -y + f_n(t)$$

Force from Harmonic Potential $V(y) = \frac{y^2}{2}$

Forcing with transfer of momentum! $\int_0^\infty f(t)\,dt = 1.$



Solving * by Laplace transform:

$$(p^2+1)\mathcal{L}[y] = \mathcal{L}[f_n]$$

$$\Rightarrow \mathcal{L}[y] = \frac{\mathcal{L}[f_n]}{p^2+1}$$

$$\Rightarrow \mathcal{L}[y] = \frac{n}{p^2+1}\int_0^{1/n} e^{-pt}\,dt$$

$$= -\frac{n}{p^2+1}\frac{e^{-p/n}}{p} + \frac{n}{(p^2+1)p}$$

$$\Rightarrow y_n(t) = \begin{cases} n(1-\cos(t)), & 0<t<1/n \\ n(1-\cos(t)) - n(1-\cos(t-1/n)), & t \geq 1/n \end{cases}$$

$$= \begin{cases} n(1-\cos(t)), & 0<t<1/n \\ -n\cos(t) + n\cos(t-1/n), & t \geq 1/n \end{cases}$$

$$= \begin{cases} n(1-\cos(t)), & 0<t<1/n \\ -n\cos(t) + n\cos(t)\cos(1/n) + n\sin(t)\sin(1/n), & t \geq 1/n \end{cases}$$

For $t \neq 0$, $\lim_{n\to\infty} y_n(t) = \sin(t)$.

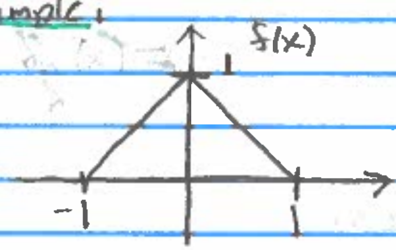
For $t = 0$, $\lim_{n\to\infty} y_n(t) = 0$.

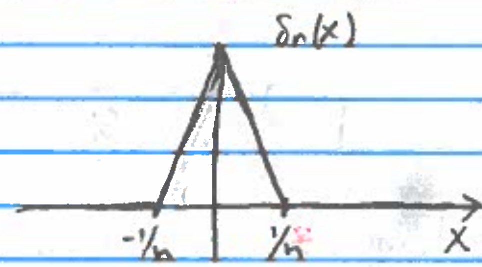
$$\Rightarrow \boxed{\lim_{n\to\infty} y_n(t) = \sin(t)}$$

Delta Sequence

1. Let $\delta_1(x) = f(x)$, where f is smooth and $\int_{-\infty}^{\infty} f(x)dx = 1$.
2. Define $\delta_n(x) = n\, f(nx)$.

Example:





$$f(x) = \begin{cases} 1-|x|, & |x|<1 \\ 0, & \text{o.w.} \end{cases}$$

$$\Rightarrow \delta_n(x) = \begin{cases} n(1-|nx|), & |x|<1/n \\ 0, & \text{o.w.} \end{cases}$$

Claim:

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\delta_n(x)\,g(x)\,dx = g(0)$$

proof:

$$\left|\int_{-\infty}^{\infty}\delta_n(x)\,g(x)\,dx - g(0)\right| = \left|\int_{-\infty}^{\infty}\delta_n(x)\left(g(x)-g(0)\right)dx\right|$$

$$= \left|\int_{-1/n}^{1/n} n\left(1-|nx|\right)\left(g(x)-g(0)\right)dx\right|$$

Mean value theorem

$$= \left|\int_{-1/n}^{1/n} n\left(1-|nx|\right)\cdot M\cdot x\, dx\right|$$

$$\le 2M\int_0^{1/n}\left(1-nx\right)dx$$

$$= 2M\left(x-\frac{nx^2}{2}\right)\Big|_0^{1/n}$$

$$\Rightarrow \lim_{n\to\infty}\left|\int_{-\infty}^{\infty}\delta_n(x)\,g(x)\,dx - g(0)\right| = 0.$$

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Delta Function:

Definition - The Dirac delta "function" satisfies

$$\int_{-\infty}^{\infty}\delta(x)\,f(x)\,dx = f(0).$$

Example:

Solve:

$$\frac{d^2y}{dx^2} + y = \delta(t-t_0)$$

$$\Rightarrow (p^2+1)\mathcal{L}[y] = \mathcal{L}[\delta(t-t_0)]$$

$$= \int_0^{\infty} e^{-pt}\,\delta(t-t_0)$$

$$= e^{-pt_0}$$

$$\Rightarrow \mathcal{L}[y] = \frac{e^{-pt_0}}{p^2+1}$$

$$\Rightarrow y(t) = \begin{cases}\sin(t-t_0), & t>t_0\\ 0 & t<t_0\end{cases}$$

Properties

1. $\delta(-x) = \delta(x)$
2. $\delta'(-x) = -\delta'(x)$
3. $\delta(ax) = \frac{1}{|a|}\delta(x)$
4. $\delta(f(x)) = \sum_i \frac{\delta(x-x_i)}{|f'(x_i)|}$ if $f(x_i)=0$, and $f'(x_i) \neq 0$.

Derivatives:

Let

$$H(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

1. $$\int_{-\infty}^{\infty} H'(x)\phi(x)\,dx = -\int_{-\infty}^{\infty} H(x)\phi'(x)\,dx$$

$$= -\int_0^{\infty} \phi'(x)\,dx$$

$$= \phi(0)$$

$$\Rightarrow H'(x) = \delta(x).$$

2. $$\int_{-\infty}^{\infty} \delta(x)'\phi(x)\,dx = -\int_{-\infty}^{\infty} \delta(x)\phi'(x) = -\phi'(0).$$

Example:

$$y'' + 4y' + 4y = \delta(t-t_0)$$

$$y(0) = 0$$

$$y'(0) = 0$$

$$(p^2+4p+4)\mathcal{L}[y] = \mathcal{L}[\delta(t-t_0)]$$

$$(p+2)^2\mathcal{L}[y] = \int_0^{\infty} \delta(t-t_0)e^{-pt}\,dt$$

$$\mathcal{L}[y] = \frac{e^{-pt_0}}{(p+2)^2}$$

$$\Rightarrow y = \begin{cases} (t-t_0)e^{-2(t-t_0)}, & t \geq t_0 \\ 0 & t < t_0 \end{cases}$$