Lecture 6: The Gamma Function, Beta Function, Erf.

$$\Gamma(p) = \int_0^\infty x^{p-1} e^{-x}\,dx. \qquad (p>0).$$

Properties:

1. $\Gamma(1) = \int_0^\infty e^{-x}\,dx = -e^{-x}\Big|_0^\infty = 1$
2. $\Gamma(2) = \int_0^\infty x e^{-x}\,dx$
$$= -xe^{-x}\Big|_0^\infty + \int_0^\infty e^{-x}\,dx$$
$$= \int_0^\infty e^{-x}\,dx$$
$$= 1.$$
3. $\Gamma(3) = \int_0^\infty x^2 e^{-x}\,dx$
$$= -x^2 e^{-x}\Big|_0^\infty + \int_0^\infty 2x e^{-x}\,dx$$
$$= 2\Gamma(2) = 2.$$

3. $\Gamma(p+1) = \int_0^\infty x^p e^{-x}\,dx$
$$= -x^p e^{-x}\Big|_0^\infty + \int_0^\infty p x^{p-1} e^{-x}\,dx$$
$$= p\Gamma(p).$$

4. If $n \in \mathbb{N}$ then
$$\Gamma(n) = (n-1)\Gamma(n-1)$$
$$= (n-1)(n-2)\Gamma(n-3)$$
$$= (n-1)!$$

Example:
$$\frac{\Gamma(4)\cdot\Gamma(3/4)}{\Gamma(7/4)} = \frac{\Gamma(4)\cdot\cancel{\Gamma(3/4)}}{3/4\,\cancel{\Gamma(3/4)}} = \frac{4}{3}\cdot 3! = 4\cdot 2! = 8.$$

More Properties:

1. $\Gamma(1/2) = \sqrt{\pi}$
2. $\Gamma(p)\Gamma(1-p) = \dfrac{\pi}{\sin(\pi p)}$

Example:

$$\Gamma(p+1) = p\,\Gamma(p)$$

$$\Rightarrow \frac{1}{p}\Gamma(p+1) = \Gamma(p)$$

$$\Rightarrow \Gamma(-3/2) = \left(\frac{-2}{3}\right)\Gamma(-1/2) = \left(\frac{-2}{3}\right)\cdot\left(\frac{-2}{1}\right)\Gamma(1/2)$$

$$\Rightarrow \Gamma(-3/2) = \frac{4}{3}\sqrt{\pi}.$$

Beta Function

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx, \quad p>0,\ q>0.$$

Example:

$$\int_0^a y^{p-1}(a-y)^{q-1}\,dy = a^{q-1}\int_0^a y^{p-1}(1-(y/a))^{q-1}\,dy$$

$$= a^q\int_0^1 (au)^{p-1}(1-u)^{q-1}\,du$$

$$= a^{q+p-1}\int_0^1 u^{p-1}(1-u)^{q-1}\,du$$

$$= a^{q+p-1}B(p,q).$$

Example:

Let $x = \sin^2\theta$

$$\Rightarrow B(p,q) = 2\int_0^{\pi/2}(\sin\theta)^{2p-1}(\cos\theta)^{2p-1}\,d\theta.$$

Example:

Let $x = y/(1+y)$

$$\Rightarrow B(p,q) = \int_0^{\infty}\frac{y^{p-1}}{(1+y)^{p+q}}\,dy$$

$$B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

proof:

$$\Gamma(p)=\int_0^\infty t^{p-1}e^{-t}dt$$
$$\Gamma(q)=\int_0^\infty t^{q-1}e^{-t}dt$$

Let $t=y^2$, $t=x^2$

$$\Rightarrow \Gamma(p)=2\int_0^\infty y^{2p-1}e^{-y^2}dy,\ \Gamma(q)=2\int_0^\infty x^{2p-1}e^{-x^2}dx$$

$$\Rightarrow \Gamma(p)\Gamma(q)=4\int_0^\infty\int_0^\infty y^{2p-1}x^{2q-1}e^{-(x^2+y^2)}dxdy$$

$$=4\int_0^\infty\int_0^{2\pi} r^{2p+2q-1}(\cos\theta)^{2p-1}(\sin\theta)^{2q-1}e^{-r^2}d\theta dr$$
$$=2B(p,q)\int_0^\infty r^{2p+2q-1}e^{-r^2}dr$$

$$=2B(p,q)\Gamma(p+q)$$

$$\Rightarrow B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

Example:

$\ddot\theta=-\sin\theta.$

$\theta(0)=\pi/2,\ \dot\theta(0)=0$

What is the period of motion?

$$\dot\theta\ddot\theta=-\dot\theta\sin\theta$$
$$\frac{d}{dt}\frac{1}{2}\dot\theta^2=\frac{d}{dt}\cos\theta$$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2}\dot\theta^2-\cos\theta\right)=0$$

$$\Rightarrow \frac{1}{2}\dot\theta^2-\cos\theta=E$$

$E = -\cos(\pi/2) = 0$

$\Rightarrow \frac{1}{2}\dot{\theta}^2 = \cos\theta$

$\Rightarrow \frac{d\theta}{dt} = \sqrt{2\cos\theta}$

$\Rightarrow T = \frac{4}{\sqrt{2}} \int_0^{\pi/2} \frac{1}{\sqrt{\cos\theta}} \, d\theta$

$\Rightarrow T = \frac{2}{\sqrt{2}} B(\tfrac{1}{2}, \tfrac{1}{4})$

Error Function

$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$

$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt = \frac{1}{2} + \frac{1}{2}\mathrm{erf}\left(\frac{x}{\sqrt{2}}\right)$.

$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt = 1 - \mathrm{erf}(x)$.

*small x.

$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \int_0^x \left(1 - t^2 + \frac{t^4}{2!} + \ldots\right) dt$

$= \frac{2}{\sqrt{\pi}} \left(x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} + \ldots\right)$