MST 205
Spring 2022
Exam #1
02/09/22

Name (Print): Key

The following rules apply:

- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Short answer questions:** Questions labeled as **"Short Answer"** can be answered by simply writing an equation or a sentence or appropriately drawing a figure. No calculations are necessary or expected for these problems.
- **Unless the question is specified as short answer, mysterious or unsupported answers might not receive full credit.** An incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

Do not write in the table to the right.

Problem	Points	Score
1	15	
2	15	
3	15	
4	15	
5	20	
6	5	
7	5	
8	10	
Total:	100	

1. (15 points) (**Short Answer**) Determine if the following statement is correct (**C**) or incorrect (**I**). Just circle **C** or **I**. No need to show any work. In order for a statement to be correct it must be true in all cases.

   C **I** If A and B are $n \times n$ matrices then $(AB)^2 = A^2B^2$.

   C **I** If A is an $n \times n$ matrix satisfying $\det(A) = 0$ then the equation $A\vec{x} = 0$ has no solutions.

   C **I** If A is an $n \times n$ matrix then $\det(A^2) = \det(A)^2$.

   C **I** If A is an $n \times n$ matrix satisfying $A^2 = 0$ then $A = 0$.

   ~~**C**~~ **I** If A is an $n \times n$ matrix satisfying $A^2 = I$ then $A = I$.

Rubric-
*3 points per problem
*No partial credit.

2. (15 points) Find all possible solutions to the following system of linear equations or state that are no solutions.

$$x + y + 2z = 9,$$
$$2x + 4y - 4z = 2,$$
$$3x + 6y - 6z = 3.$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 2 & 4 & -4 & 2 \\ 3 & 6 & -6 & 3 \end{array}\right] \begin{array}{l} \\ -2R1 \\ -3R1 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -8 & -16 \\ 0 & 3 & -12 & -24 \end{array}\right] \begin{array}{l} \\ /2 \\ /3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -4 & -8 \\ 0 & 1 & -4 & -8 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -4 & -8 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$z =$ anything
$y = -8 + 4z$
$x = 9 - 2z - y$

$\Rightarrow$ $z =$ anything
$y = -8 + 4z$
$z = 17 - 6z$

Rubric
*10 points row reduction
* -1 point minor mistake (90%)
* -3 points 2 minor mistake (70%)
*-5 points 3 or more minor mistakes (50%)
*-10 points no understanding. (0%)
*2 points minimum for just setting up matrix.

*5 points solution interpretation
*Based off row reduction.
*No partial credit for wrong interpretation.
*-1 point for minor mistake solving for x, y.

3. (15 points) Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}.$$

Compute the following or state that the computation is impossible.

(a) (5 points) $-3(C - 2B)$

$$-3(C-2B) = -3\left(\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 8 & 6 \\ 4 & 2 \end{bmatrix}\right) = -3\left(\begin{bmatrix} -7 & -6 \\ -2 & 1 \end{bmatrix}\right) = \begin{bmatrix} 21 & 18 \\ 6 & -3 \end{bmatrix}.$$

Rubric

Each part

* -1 point for small mistake (8

* -3 points for two or more small mistakes but shows understanding (40%)

* -5 No understanding.

$\Downarrow$

No partial credit for C.

(b) (5 points) AB

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ 20 & 13 \\ 2 & 1 \end{bmatrix}$$

(c) (5 points) BA.

Not possible.

4. (15 points) Find the inverse of the following matrix or state why it is not invertible.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]\begin{array}{c} \\ -R1 \\ \\ \end{array} \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]\begin{array}{c} \\ \\ -R2 \end{array} \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right]\begin{array}{c} -R3 \\ +R3 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right]$$

Therefore,

$$A^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

Rubric -

* -1 small mistake
* -3 two small mistakes
* -5 three or small mistakes

} correct setup.

* -15 No understanding.

5. (20 points) Calculate the determinants of the following matrices where a, b, c, d, e, f are constants.

(a) (5 points) $A = \begin{bmatrix} -3 & 0 & 7 \\ 2 & 5 & 0 \\ -1 & 0 & 5 \end{bmatrix}$

$\det(A) = 5\det\left(\begin{bmatrix} -3 & 7 \\ -1 & 5 \end{bmatrix}\right) = 5(-15+7) = -40$

(b) (5 points) $B = \begin{bmatrix} 1 & a & a^2 \\ 1 & a & a^2 \\ 1 & a & a^2 \end{bmatrix}$

$\det(B) = 0$ since colums are linearly dependent.

(c) (5 points) $C = \begin{bmatrix} 1 & \sqrt{2} & \pi \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$

$\det(C) = 6.$

Robric

-1 small mistake

-2 two or more small mistakes

-5 No understanding.

(d) (5 points) $D = \begin{bmatrix} 0 & a & 0 & 0 \\ b & 0 & c & 0 \\ 0 & d & 0 & e \\ 0 & 0 & f & 0 \end{bmatrix}$.

$\det(D) = -a\det\left(\begin{bmatrix} b & c & 0 \\ 0 & 0 & e \\ 0 & f & 0 \end{bmatrix}\right)$

$= -a\cdot(-e)\det\left(\begin{bmatrix} b & c \\ 0 & f \end{bmatrix}\right)$

$= aebf.$

6. (5 points) **Short Answer:** Suppose A in an $n \times n$ matrix. Write down the fundamental definition of what it means for an $n \times n$ matrix B to be an inverse of A.

$$\left.\begin{array}{l} BA = I \\ AB = I \end{array}\right\}$$

-1 point if they only have one equation

*There are lots of answers that students might provide. No partial credit if completely wrong.

7. (5 points) Suppose A and B are $n \times n$ invertible matrices. Show that

$$(AB)^{-1} = B^{-1}A^{-1}.$$

$$B^{-1}A^{-1} \cdot AB = B^{-1} I \cdot B = B^{-1} \cdot B = I$$

*We will see when we get here...

8. (10 points) For what values of a does the following system of equations have zero, one, or infinitely many solutions?

$$x_1 + x_2 + x_3 = 4,$$
$$x_3 = 2,$$
$$(a^2 - 4)x_3 = a - 2.$$

$X_3 = 2$

$\Rightarrow (a^2-4)2 = a-2$

If $a=2$ we get

$X_1 + X_2 + 2 = 4$

$\Rightarrow X_1 = 2 - X_2$, i.e. infinite number of solutions.

Otherwise we must have

$2(a-2)(a+2) = a-2$

$\Rightarrow 2a+4 = 1$

$\Rightarrow a = -3/2$.

If $a=2$ or $a=-3/2$ we have infinite number of solutions

Otherwise we have none.