**MST 205**
**Spring 2022**
**Exam #2**
**03/16/22**

**Name (Print):** Key

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The following rules apply:

- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Short answer questions:** Questions labeled as **"Short Answer"** can be answered by simply writing an equation or a sentence or appropriately drawing a figure. No calculations are necessary or expected for these problems.
- **Unless the question is specified as short answer, mysterious or unsupported answers might not receive full credit.** An incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

Do not write in the table to the right.

| Problem | Points | Score |
|---|---|---|
| 1 | 10 | |
| 2 | 15 | |
| 3 | 15 | |
| 4 | 10 | |
| 5 | 10 | |
| 6 | 15 | |
| 7 | 15 | |
| 8 | 10 | |
| Total: | 100 | |

1. (10 points) (**Short Answer**) Determine if the following statement is correct (**C**) or incorrect (**I**). Just circle **C** or **I**. No need to show any work. In order for a statement to be correct it must be true in all cases.

In these problems, $F(-1,1)$ denotes the vector space of continuous functions defined on $(-1,1)$ with the standard operations of addition and multiplication.

**Hint:** If the answer to these questions does not come quickly just move on and come back later.

**C** **I** The set of all pairs of real numbers of the form $(0, y)$ with the standard operations of addition and multiplication is a subspace of $\mathbb{R}^2$.

**C** **I** The set of all pairs of real numbers of the form $(x, -x)$ with the standard operations of addition and multiplication is a subspace of $\mathbb{R}^2$.

**C** **I** The set of all pairs of real numbers of the form $(x, x^2)$ with the standard operations of addition and multiplication is a subspace of $\mathbb{R}^2$.

**C** **I** The set of functions $f(x)$ such that $f(x) \geq 0$ for all $x$ is a subspace of $F(-1,1)$.

**C** **I** The set of functions $f(x)$ such that $f(-1) = f(1)$ is a subspace of $F(-1,1)$.

**C** **I** The set of functions $f(x)$ such that $\int_{-1}^{1} f(x)dx = 0$ is a subspace of $F(-1,1)$.

**C** **I** If $A$ is $n \times n$ matrix satisfying $\det(A) \neq 0$ then the dimension of $CS(A))$ is equal to $n$.

**C** **I** If $A$ is $n \times n$ matrix satisfying $\det(A) \neq 0$ then the dimension of $NS(A)$ is not equal to 0.

**C** **I** If $A$ is an $m \times n$ matrix with $m > n$, then the rows of $A$ are linearly dependent.

**C** **I** The set of quadratic polynomials with no zeros is a subspace of $P_2$.

2. (15 points)

(a) (5 points) Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be vectors in a vector space $V$. Write down what it means for $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ to form a linearly independent set.

The only solution to the equation

$$c_1\vec{v}_1 + \ldots + c_n\vec{v}_n = 0$$

is $c_1 = c_2 = \ldots = c_n = 0$

(b) (10 points) Determine whether the given matrices are linearly dependent or linearly independent

$$\left\{ \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}.$$

$$c_1\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} + c_2\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c_3\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} c_1 + c_3 &= 0 \\ c_2 + c_3 &= 0 \\ -c_1 + c_2 + c_3 &= 0 \end{aligned} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 1 & 1 & 0 \end{array}\right] +R1$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] -R2 \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \Rightarrow c_1 = c_2 = c_3 = 0$$

Therefore, linearly independent.

3. (15 points)

(a) (5 points) Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be vectors in a vector space $V$. Write down what it means for a vector $\mathbf{w} \in V$ to lie in the span of $\{\mathbf{v}_1, \ldots \mathbf{v}_n\}$. Equivalently, you can write down a definition for the subspace $W = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$.

$\vec{W} \in \text{span}\{\vec{V}_1, \ldots, \vec{V}_n\}$

there exists $k_1, \ldots, k_n \in \mathbb{R}$ such that

$\vec{W} = k_1\vec{V}_1 + \ldots + k_n\vec{V}_n$

(b) (10 points) For what values of $c$ does the vector

$$\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ c \end{bmatrix}$$

lie in the span of the following vectors

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -5 \\ 14 \\ 7 \end{bmatrix} \right\}.$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -5 & 1 \\ -2 & 1 & 14 & 1 \\ 1 & 3 & 7 & c \end{array}\right] \begin{array}{l} \\ +2R1 \\ -R1 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -5 & 1 \\ 0 & 1 & 4 & 3 \\ 0 & 3 & 12 & c-1 \end{array}\right] \begin{array}{l} \\ \\ -3R2 \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -5 & 1 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & c-10 \end{array}\right]$$

$\Rightarrow c = 10.$

4. (10 points)

(a) (5 points) Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be vectors in a vector space $V$. Write down what it means for $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ to form a basis for $V$.

$V = \text{span}\{\vec{v}_1, \ldots, \vec{v}_n\}$
and $\{\vec{v}_1, \ldots, \vec{v}_n\}$ are linearly independent.

(b) (5 points) Determine if the polynomials $p_1(x) = x^2 + x + 2$, $p_2(x) = x^2 + 2x + 1$, and $p_3(x) = 2x^2 + 5x + 1$ form a basis for $P_2$.

$$c_1(x^2+x+2) + c_2(x^2+2x+1) + c_3(2x^2+5x+1) = 0$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 1 & 2 & 5 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right] \begin{array}{l} \\ -R1 \\ -2R1 \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & -1 & -3 & 0 \end{array}\right]$$

Does not determine a basis.

5. (10 points) Let $\alpha$ be the basis given by

$$\left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix} \right\}.$$

(a) (5 points) Find $[v]_\alpha$ if

$$v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 1 \\ -2 & 4 & 1 \end{array}\right] {+2R1} \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 6 & 3 \end{array}\right]$$

$$\Rightarrow c_2 = 1/2$$

$$c_1 = 1/2$$

$$[v]_\alpha = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

(b) (5 points) Find $w$ if

$$[w]_\alpha = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$$[w]_\alpha = 1 \cdot \begin{bmatrix} 1 \\ -2 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

6. (15 points) If

$$A = \begin{array}{c} \begin{bmatrix} 1 & 2 & -1 & 3 & 0 \\ 1 & 1 & 0 & 4 & 1 \\ 1 & 4 & -3 & 1 & -2 \end{bmatrix} \end{array} \begin{array}{l} \\ -R1 \\ -R1 \end{array}$$

find a basis for $NS(A)$, $RS(A)$, and determine the rank of $A$.

$$\left[\begin{array}{ccccc|c} 1 & 2 & -1 & 3 & 0 & 0 \\ 0 & -1 & 1 & 1 & 1 & 0 \\ 0 & 2 & -2 & -2 & -2 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccccc|c} 1 & 2 & -1 & 3 & 0 & 0 \\ 0 & 1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right]$$

$$x_2 = x_3 + x_4 + x_5$$

$$x_1 + 2(x_3 + x_4 + x_5) - x_3 + 3x_4 = 0$$

$$x_1 + x_3 + 5x_4 + 2x_5 = 0$$

$$x_1 = -x_3 - 5x_4 - 2x_5$$

$$\begin{bmatrix} -x_3 - 5x_4 - 2x_5 \\ x_3 + x_4 + x_5 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

7. (15 points) Suppose $A$ is a $5 \times 3$ matrix.

   (a) (5 points) **(Short Answer:)** What is the largest possible value for the dimension of the columnspace of $A$?

   3

   (b) (5 points) **(Short Answer:)** What is the largest possible value for the dimension of the rowspace of $A$?

   3

   (c) (5 points) **(Short Answer:)** What is the smallest possible value for the dimension of the nullspace of $A$?

   0

8. (10 points) Using the Wronskian, determine if the functions $f(x) = 1/x$ and $g(x) = x$ are linearly independent on the interval $(0, \infty)$.

$f'(x) = -1/x^2$

$g'(x) = 1$

$\det \begin{bmatrix} 1/x & x \\ -1/x^2 & 1 \end{bmatrix} = \frac{1}{x} + \frac{1}{x} = \frac{2}{x} \neq 0.$