

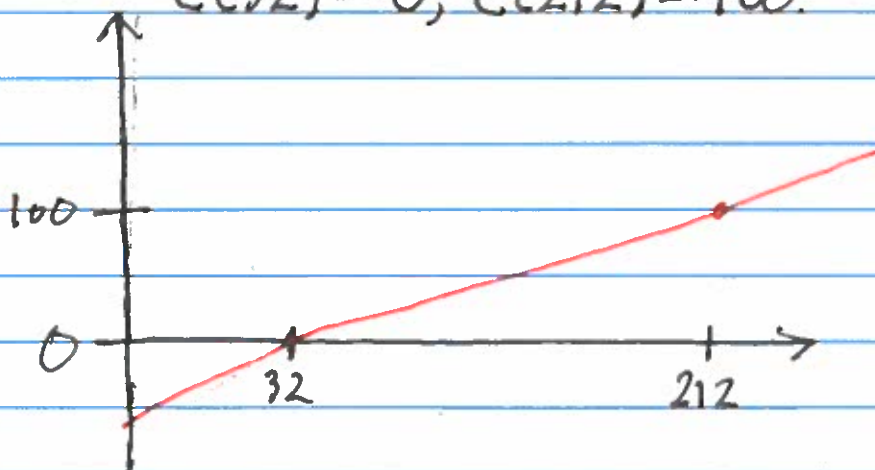
## Section 1.1: Systems of Linear Equations

**Example:**

Convert  $50^{\circ}\text{F}$  to degrees celsius. Define  $C(F)$   $\leftarrow$  function takes in  $^{\circ}\text{F}$  and outputs  $^{\circ}\text{C}$ .

What we know:

$$C(32) = 0, C(212) = 100.$$



$C(F) = mF + b$ , What is  $C(50)$ ??

$$* C(32) = m32 + b = 0$$

$$* C(212) = m212 + b = 100$$

We need to solve:

$$32m + b = 0$$

$$212m + b = 100$$

$\rightarrow$  System of two linear equations.

Method 1:

$$32m + b = 0$$

$$\Rightarrow b = -32m$$

$$\Rightarrow 212m - 32m = 100$$

$$\Rightarrow 180m = 100$$

$$\Rightarrow m = \frac{100}{180} = \frac{5}{9} \Rightarrow C(F) = \frac{5}{9}(F - 32)$$

$$b = -32 \cdot \frac{5}{9} \quad C(50) = \frac{5}{9} \cdot 18 = 10$$

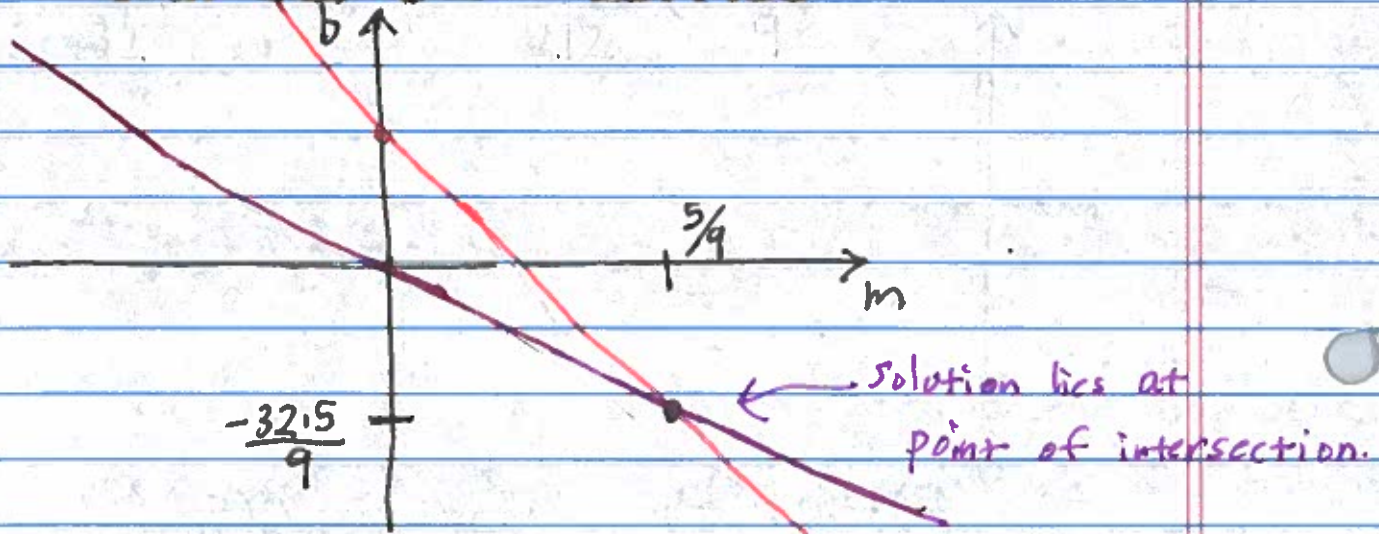
### Method 2:

$$\begin{array}{l} 32m + b = 0 \\ 212m + b = 100 \end{array} \quad \begin{array}{l} \text{Subtracting first from} \\ \text{Second} \end{array} \Rightarrow 180m = 100 \Rightarrow m = \frac{5}{9}$$

$$\text{Therefore, } 32 \cdot \frac{5}{9} + b = 0 \Rightarrow b = -\frac{32 \cdot 5}{9}$$

### Method 3:

$$b = -32m \text{ and } b = -212m + 100$$



### Example:

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{array} \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{System of two linear equations.}$$

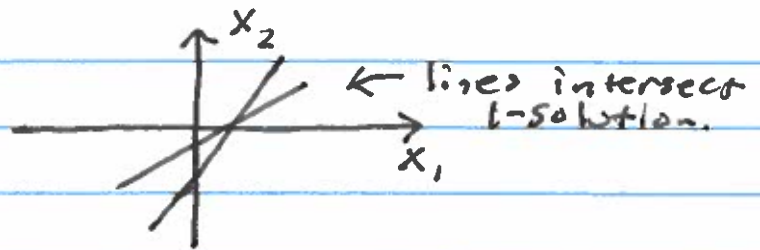
$$\Rightarrow x_2 = \frac{-a_{11}}{a_{12}}x_1 + \frac{b_1}{a_{12}} \quad \begin{array}{l} \text{slope } m_1 \\ x_2\text{-intercept} \end{array}$$

$$x_2 = \frac{-a_{21}}{a_{22}}x_1 + \frac{b_2}{a_{22}} \quad \begin{array}{l} x_2\text{-intercept} \\ \text{slope } m_2 \end{array}$$



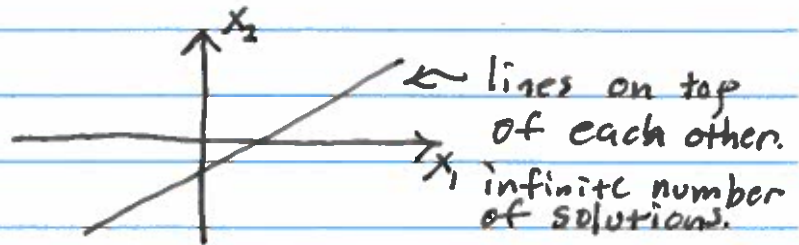
Case 1:

$$\frac{-a_{11}}{a_{12}} \neq \frac{-a_{21}}{a_{22}}$$



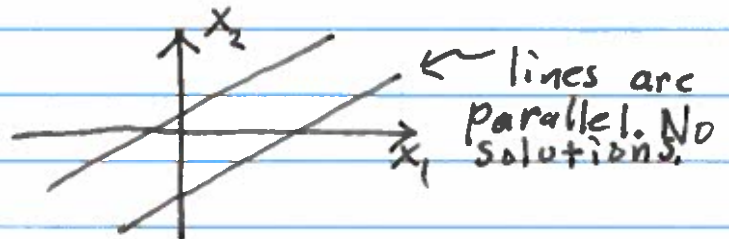
Case 2:

$$\frac{-a_{11}}{a_{12}} = \frac{-a_{21}}{a_{22}}$$
$$b_1/a_1 = b_2/a_2$$



Case 3:

$$\frac{-a_{11}}{a_{12}} = \frac{-a_{21}}{a_{22}}$$
$$b_1/a_1 \neq b_2/a_2$$



Example:

Solve the following system of equations!

(R1)  $x + y - z = 0$                        $x + y - z = 0$

(R2)  $2x + 3y - 2z = 6$      $-2R1 \rightarrow (3y - 2y) + (-2z + 2z) = 6$

(R3)  $x + 2y + 2z = 10$      $-R1 \quad (2y - y) + (2z + z) = 10$

$$\Rightarrow x + y - z = 0$$

$$y = 6$$

$$y + 3z = 10 \quad -R2$$

$$\Rightarrow x + y - z = 0$$

$$y = 6$$

$$3z = 4$$

$$\Rightarrow z = 4/3$$

$$y = 6$$

$$x + 6 - 4/3 = 0$$

$$3x + 18 - 4 = 0$$

$$x = -14/3$$

Example:

$$x_1 + x_2 - x_3 + 2x_4 = 1$$

$$x_1 + x_2 - x_3 - x_4 = -1$$

$$x_1 + 2x_2 + x_3 + 2x_4 = -1$$

$$2x_1 + 2x_2 + x_3 + x_4 = 2$$

← The variables are irrelevant.

$$\begin{array}{l} -R1 \\ -R1 \\ -2R1 \end{array} \left[ \begin{array}{cccc|c} 1 & 1 & -1 & 2 & 1 \\ 1 & 1 & -1 & -1 & -1 \\ 1 & 2 & 1 & 2 & -1 \\ 2 & 2 & 1 & 1 & 2 \end{array} \right] \left. \vphantom{\begin{array}{l} -R1 \\ -R1 \\ -2R1 \end{array}} \right\} \text{Augmented Matrix}$$

Coefficient Matrix.

Goal: eliminate coefficients to isolate variables.

$$\begin{array}{l} \updownarrow \\ \updownarrow \end{array} \left[ \begin{array}{cccc|c} 1 & 1 & -1 & 2 & 1 \\ 0 & 0 & 0 & -3 & -2 \\ 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 3 & -3 & 0 \end{array} \right] \rightarrow \left[ \begin{array}{cccc|c} 1 & 1 & -1 & 2 & 1 \\ 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & -3 & -2 \\ 0 & 0 & 3 & -3 & 0 \end{array} \right] \updownarrow$$

$$\rightarrow \left[ \begin{array}{cccc|c} 1 & 1 & -1 & 2 & 1 \\ 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 3 & -3 & 0 \\ 0 & 0 & 0 & -3 & 0 \end{array} \right] \begin{array}{l} R3/3 \\ R4/-3 \end{array} \rightarrow \left[ \begin{array}{cccc|c} 1 & 1 & -1 & 2 & 1 \\ 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right]$$

$$\Rightarrow x_1 + x_2 - x_3 + 2x_4 = 1$$

$$x_2 + 2x_3 = -2$$

$$x_3 - x_4 = 0$$

$$x_4 = 0$$

$$\Rightarrow x_4 = 0, x_3 = 0, x_2 = -2, x_1 = 3.$$



### Example:

$$\begin{aligned} 2x + 3y - 4z &= 3 \\ 2x + 3y - 2z &= 3 \\ 4x + 6y - 8z &= 6 \end{aligned} \rightarrow \begin{bmatrix} 2 & 3 & -4 & 3 \\ 2 & 3 & -2 & 3 \\ 4 & 6 & -8 & 6 \end{bmatrix} \begin{array}{l} \\ -R1 \\ -2R1 \end{array}$$

$$\rightarrow \begin{bmatrix} 2 & 3 & -4 & 3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$2x + 3y - 4z = 3$$

$$2z = 0$$

$0 = 0 \rightarrow$  No useful information.

$$\Rightarrow z = 0,$$

$$2x + 3y = 3$$

$$\Rightarrow y = -\frac{2}{3}x + 1$$

### Example:

Determine Conditions on  $a, b, c$  so that the following system of equations has a solution:

$$\begin{aligned} 2x - y + 3z &= a \\ x - 3y + 2z &= b \\ x + 2y + z &= c \end{aligned} \Rightarrow \begin{bmatrix} 2 & -1 & 3 & a \\ 1 & -3 & 2 & b \\ 1 & 2 & 1 & c \end{bmatrix} \begin{array}{l} -2R2 \\ \\ -R2 \end{array} \Rightarrow \begin{bmatrix} 0 & 5 & -1 & a-2b \\ 1 & -3 & 2 & b \\ 0 & 5 & -1 & c-b \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 5 & -1 & a-2b \\ 1 & -3 & 2 & b \\ 0 & 0 & 0 & c+b-a \end{bmatrix}$$

For these equations to be consistent we need  $c+b-a=0$ .

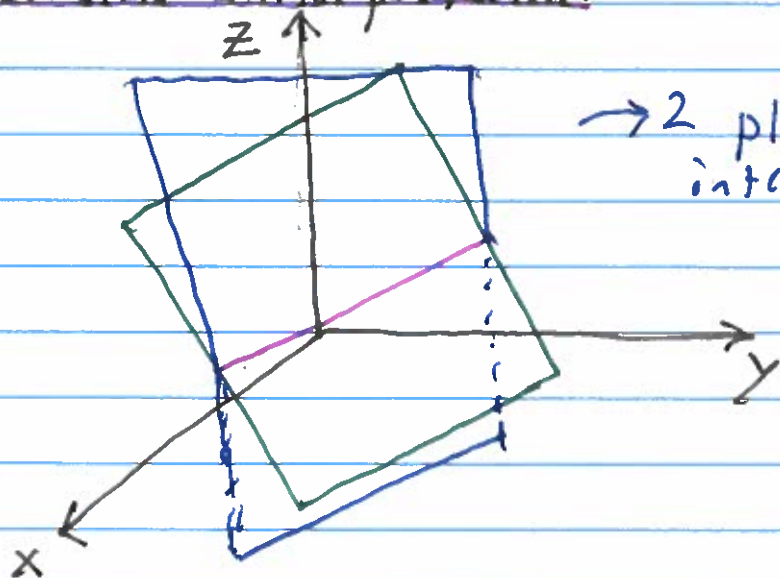
### Example:

Does the following system have nontrivial solutions?

$$\begin{aligned} x - y + z &= 0 \\ 2x + y + 2z &= 0 \\ 3x - 5y + 3z &= 0 \end{aligned} \Rightarrow \begin{bmatrix} 1 & -1 & 1 & 0 \\ 2 & 1 & 2 & 0 \\ 3 & -5 & 3 & 0 \end{bmatrix} \begin{array}{l} \\ -2R_1 \\ -3R_1 \end{array}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{l} y = 0 \\ z = \text{anything} \\ x = -z \end{array}$$

Geometric Interpretation:



→ 2 planes intersect in a line.