Section 2.5: Wronskians.

Example:

Are the functions $1+x^2$, $1-x^2$, $1+x$ linearly independent?

$$c_1(1+x^2)+c_2(1-x^2)+c_3(1+x)=0$$

$$\Rightarrow c_1+c_2+c_3=0$$

$$c_3=0$$

$$c_1-c_2=0$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ $\Rightarrow$ The only solution is $c_1=c_2=c_3=0$.

$\Rightarrow$ linearly independent.

Example:

Are the functions $\{\sin^2(x), \cos^2(x), 5\}$ linearly independent

$$c_1\sin^2(x)+c_2\cos^2(x)+5c_3=0$$

$$\Rightarrow c_1(1-\cos^2(x))+c_2\cos^2(x)+5c_3=0$$

$$\Rightarrow (c_1+5c_3)+(c_2-c_1)\cos^2(x)=0$$

c_2 = free variable

$c_1=c_2$

$c_3=-c_2/5$.

Linearly dependent.

Idea:

Set up n-equations by taking derivatives:

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0$$
$$c_1 f_1'(x) + c_2 f_2'(x) + c_3 f_3'(x) = 0$$
$$c_1 f_1''(x) + c_2 f_2''(x) + c_3 f_3''(x) = 0$$

$$W(x) = \det\left(\begin{bmatrix} f_1(x) & f_2(x) & f_3(x) \\ f_1'(x) & f_2'(x) & f_3'(x) \\ f_1''(x) & f_2''(x) & f_3''(x) \end{bmatrix}\right)$$

If $W(x) \neq 0$ for all $x \Rightarrow$ linearly independant.

Example:

$f_1(x) = x$

$f_2(x) = \sin(x)$

$$W(x) = \det\left(\begin{bmatrix} x & \sin(x) \\ 1 & \cos(x) \end{bmatrix}\right) = x\cos(x) - \sin(x) \neq 0.$$

$\Rightarrow$ linearly independent.

Example:

$f_1(x) = \sin^2(x)$, $f_2(x) = \cos^2(x)$, $f_3 = 5$

$$W(x) = \det\left(\begin{bmatrix} \sin^2(x) & \cos^2(x) & 5 \\ 2\sin(x)\cos(x) & -2\sin(x)\cos(x) & 0 \\ 2\cos^2(x) - 2\sin^2(x) & -2\cos^2(x) + 2\sin^2(x) & 0 \end{bmatrix}\right)$$

$= 0$

$\Rightarrow$ linearly dependent.

Example:

$f_1 = 1,\ f_2 = e^x,\ f_3 = e^{2x}$

$$W(x) = \det\left(\begin{bmatrix} 1 & e^x & e^{2x} \\ 0 & e^x & 2e^{2x} \\ 0 & e^x & 4e^{2x} \end{bmatrix}\right)$$

$$= 2e^{3x} \neq 0$$

$\Rightarrow$ linearly independent.