

Section 4.2: Homogeneous Constant Coefficient Systems.

Example:

$$y''' - 4y' = 0$$

Guess:

$$y = e^{\lambda x}$$

$$\lambda^3 - 4\lambda = 0$$

$$\Rightarrow \lambda = 0, \lambda = \pm 1$$

General Solution is

$$\begin{aligned} y(x) &= c_1 e^{0x} + c_2 e^x + c_3 e^{-x} \\ &= c_1 + c_2 e^x + c_3 e^{-x}. \end{aligned}$$

Example:

$$y'' + 5y' + 6y = 0$$

$$y(0) = 2$$

$$y'(0) = 3$$

$$y = e^{\lambda x}$$

$$\Rightarrow \lambda^2 + 5\lambda + 6 = 0$$

$$(\lambda + 3)(\lambda + 2) = 0$$

$$\Rightarrow \lambda = -3, -2$$

$$y(x) = c_1 e^{-3x} + c_2 e^{-2x}$$

$$y(0) = c_1 + c_2 = 2$$

$$y'(0) = -3c_1 - 2c_2 = 3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 \\ -3 & -2 & 3 \end{array} \right] \xrightarrow{+3R_1} \left[\begin{array}{ccc|c} 1 & 1 & 2 \\ 0 & 1 & 9 \end{array} \right]$$

$$c_2 = 9 \Rightarrow y(x) = -7e^{-3x} + 9e^{-2x}.$$

$$c_1 = -7$$

Reduction of Order:

Consider the equation:

$$y'' + p(x)y' + q(x)y = 0$$

with solution $y_1(x)$. Let

$$y_2(x) = v(x)y_1(x)$$

$$\Rightarrow y_2'(x) = v'y_1 + vy_1'$$

$$y_2'' = v''y_1 + 2v'y_1' + vy_1''$$

$$\Rightarrow v''y_1 + 2v'y_1' + vy_1'' + p(x)(v'y_1 + vy_1') + q(x)v \cdot y_1 = 0$$

$$\Rightarrow v''y_1 + 2v'y_1' + v(y_1'' + p(x)y_1' + q(x)y_1) + p(x) \cdot v'y_1 = 0$$

$$\Rightarrow y_1 v'' + (2y_1' + py_1)v' = 0$$

↓
First order equation in v'

Example:

$$y'' - 2y' + y = 0$$

Guess

$$y = e^{\lambda x}$$

$$\Rightarrow \lambda^2 - 2\lambda + 1 = 0$$

$$\Rightarrow (\lambda - 1)^2 = 0$$

$$\Rightarrow \lambda = 1$$

One solution is

$$y_1 = e^x$$

Guess another solution of the form

$$y_2 = v(x)e^x$$

$$\Rightarrow y_2' = v'e^x + ve^x$$

$$y_2'' = v''e^x + 2v'e^x + ve^x$$

$$\Rightarrow v''e^x + 2v'e^x + ve^x - 2v'e^x - 3ve^x + ve^x = 0$$

$$\Rightarrow v''e^x = 0$$

$$\Rightarrow v(x) = cx + d$$

General solution is

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

$$= c_1 e^x + c_2 (c + dx)e^x$$

$$= c_1 e^x + c_2 x e^x.$$

Example:

Given $y_1 = x^{-1}$ is a solution of

$$2x^2 y'' + 3xy' - y = 0$$

find a second linearly ind. solution.

$$y_2 = v x^{-1}$$

$$y_2' = \frac{-1}{x^2} v + v' x^{-1}$$

$$y_2'' = \frac{2}{x^3} v - \frac{1}{x^2} v' + v'' x^{-1} - \frac{v'}{x^2}$$

$$\Rightarrow \frac{4v}{x} - 4v' + v'' x - \frac{3v}{x} + 3v' - \frac{v'}{x} = 0$$

$$xv'' - v' = 0$$

$$\text{Let } u = \frac{dv}{dt}$$

$$\Rightarrow x \frac{du}{dx} = u$$

$$\Rightarrow \int \frac{1}{u} du = \int \frac{1}{x} dx$$

$$\ln(u) = \ln|x| + c$$

$$\Rightarrow u = ct$$

$$\Rightarrow v = ct^2 + d$$

Therefore,

$$y_2(t) = v \cdot \frac{1}{t}$$

$$= \frac{ct + d}{t}$$

The two linearly dependent solutions are

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

$$= \frac{c_1 + c_2 t}{t}$$

Complex Numbers and Complex Roots

$$z = x + yi, \quad z \in \mathbb{C} \rightarrow \text{Complex numbers (Vector space)}$$

\uparrow real # \uparrow real # $i^2 = -1$

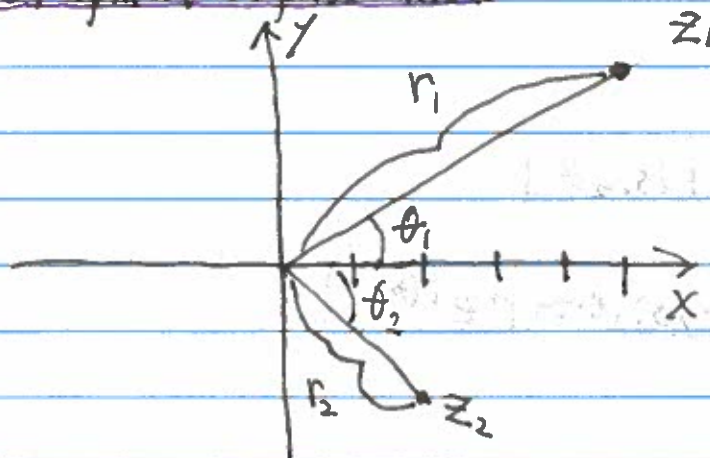
Example:

$$z_1 = 5 + 3i, \quad z_2 = 2 - 2i$$
$$- z_1 \cdot z_2 = (5 + 3i)(2 - 2i)$$
$$= 10 + 6i - 10i + 6$$
$$= 16 - 4i$$

$$- z_1 + z_2 = 7 + i$$

$$- \frac{z_1}{z_2} = \frac{5 + 3i}{2 - 2i} \cdot \frac{(2 + 2i)}{(2 + 2i)} = \frac{10 + 14i - 6}{8} = \frac{4 + 14i}{8} = \frac{1 + 7i}{2}$$

Graphical Representation:



$$r_1 = \sqrt{25 + 9} = \sqrt{34}, \quad \tan \theta_1 = \frac{3}{5}$$

$$r_2 = \sqrt{8} = 2\sqrt{2}, \quad \tan \theta_2 = -\frac{2}{2} \Rightarrow \theta_2 = -\frac{\pi}{4}$$

$$z_1 = r_1 \cos \theta_1 + r_1 \sin \theta_1 i$$

$$z_2 = r_2 \cos \theta_2 + r_2 \sin \theta_2 i$$

Euler's Formula:

$$1. e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2} + \frac{(i\theta)^3}{6} + \dots$$
$$= 1 + i\theta - \frac{\theta^2}{2} - \frac{i\theta^3}{6} + \dots$$

$$2. \cos(\theta) = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} + \dots$$

$$3. \sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots$$

$$\Rightarrow \cos\theta + i\sin\theta = 1 + i\theta - \frac{\theta^2}{2} - \frac{i\theta^3}{6} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} + \dots$$
$$= e^{i\theta}$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$z = x + iy = r\cos\theta + i r\sin\theta = r e^{i\theta}$$

Example:

1. $y'' + 4y = 0$

Guess:

$$y = e^{\lambda x}$$

$$y' = \lambda e^{\lambda x}$$

$$y'' = \lambda^2 e^{\lambda x}$$

$$\Rightarrow \lambda^2 + 4 = 0$$

$$\Rightarrow \lambda = \pm 2i$$

$$y = c_1 e^{2ix} + c_2 e^{-2ix}$$

$$y = c_1 (\cos(2x) + i \sin(2x)) + c_2 (\cos(2x) - i \sin(2x))$$

$$= c_1 \cos(2x) + c_2 \sin(2x).$$

2. $2y'' + 8y' + 26y = 0$

$$y'' + 4y' + 13y = 0$$

Guess:

$$y = e^{\lambda x}$$

$$\Rightarrow \lambda^2 + 4\lambda + 13 = 0$$

$$\lambda = \frac{-4 \pm \sqrt{16 - 4 \cdot 13}}{2}$$

$$= \frac{-4 \pm 2\sqrt{4 - 13}}{2}$$

$$= -2 \pm 3i$$

$$y = c_1 e^{(-2+3i)x} + c_2 e^{(-2-3i)x}$$

$$= e^{-2x} (c_1 e^{3ix} + c_2 e^{-3ix})$$

$$= c_1 e^{-2x} \cos(3x) + c_2 e^{-2x} \sin(3x).$$