Section 5.1: Linear Transformation:

$$f: X \to Y$$

is <u>linear tranformation</u> if

- $f(u+v) = f(u) + f(v)$
- $f(cu) = cf(u)$

Notation-

- X = <u>domain</u> of f
- Y = <u>codomain</u>
- $\{f(x) : x \in X\}$ = <u>range</u> of f.

Example:

1. $T: \mathbb{R}^3 \to \mathbb{R}^2$ defined by:

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y-z \\ x+2y+z \end{bmatrix}$$

$$\Rightarrow T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

is <u>linear</u> since

$$A(u+v) = Au + Av$$
$$A(cu) = cAu.$$

2. $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x^2 \\ x+y \end{bmatrix}$$

is not linear since

$$T\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + T\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1^2 + x_2^2 \\ x_1 + x_2 + y_1 + y_2 \end{bmatrix}$$

$$T\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix} = \begin{bmatrix} (x_1+x_2)^2 \\ x_1 + x_2 + y_1 + y_2 \end{bmatrix} = \begin{bmatrix} x_1^2 + 2x_1x_2 + x_2^2 \\ x_1 + x_2 + y_1 + y_2 \end{bmatrix}$$

3. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+1 \\ y \end{bmatrix}$

is not linear since

$$T\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \end{bmatrix} = \begin{bmatrix} x_1+x_2+1 \\ y_1+y_2 \end{bmatrix}$$

$$T\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + T\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1+1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2+1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1+x_2+2 \\ y_1+y_2 \end{bmatrix}$$

4. $\frac{d}{dx}$ is a linear operator

5. $\int_a^b f(x)\,dx$ is a linear operator.

Example:

1. Suppose $T: \mathbb{R}^3 \to \mathbb{R}^3$ satisfies

$$T\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \quad T\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \quad T\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

what is $T\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$?

$$T\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = T\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2T\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 3T\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 11 \\ 14 \end{bmatrix}.$$

(You just need to know the action on a basis!)

2. Suppose $T: \mathbb{R}^2 \to \mathbb{R}^2$ satisfies

$$T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

what is $T\begin{bmatrix} 0 \\ 1 \end{bmatrix}$?

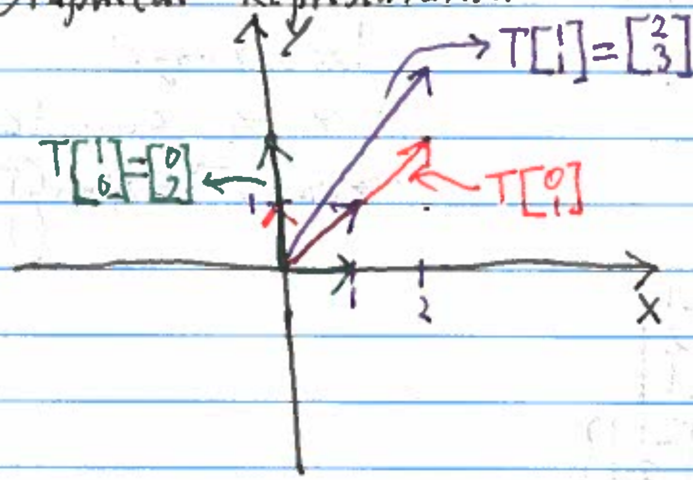
$$\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}-\begin{bmatrix}1\\0\end{bmatrix}$$

$$\Rightarrow T\begin{bmatrix}0\\1\end{bmatrix}=T\begin{bmatrix}1\\1\end{bmatrix}-T\begin{bmatrix}1\\0\end{bmatrix}$$

$$=\begin{bmatrix}2\\3\end{bmatrix}-\begin{bmatrix}0\\2\end{bmatrix}$$

$$=\begin{bmatrix}2\\2\end{bmatrix}$$

Graphical Representation.



<u>Def</u>

- $\mathrm{Ker}(T)=\{v\in V : Tv=0\}$
- $\mathrm{ran}(T)=\{Tv : v\in \mathrm{Dom}(T)\}$.

<u>Example:</u>

Find the kernel and range of $T:\mathbb{R}^3\to\mathbb{R}^2$ defined by:

$$T\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1&1&-1\\1&2&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}.$$

If $\vec{x}\in\ker(T)$ then +

$$\begin{bmatrix}1&1&-1\\1&2&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}\Rightarrow\left[\begin{array}{ccc|c}1&1&-1&0\\1&2&1&0\end{array}\right]$$

If $y\in\mathrm{range}(T)$ then there exists $\vec{x}$ such that

$$\begin{bmatrix}1&1&-1\\1&2&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}y_1\\y_2\end{bmatrix}\Rightarrow\left[\begin{array}{ccc|c}1&1&-1&y_1\\1&2&1&y_2\end{array}\right]$$

Row reduce all at once:

$$\left[\begin{array}{ccc|c|c} 1 & 1 & -1 & 0 & y_1 \\ 1 & 2 & 1 & 0 & y_2 \end{array}\right] - R1$$

$$\Rightarrow \left[\begin{array}{ccc|c|c} 1 & 1 & -1 & 0 & y_1 \\ 0 & -1 & 2 & 0 & y_2 - y_1 \end{array}\right]$$

For kernel:

$y = 2z$

$x + y - z = 0$

$x = -z$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -z \\ 2z \\ z \end{bmatrix} = z\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$$\Rightarrow \ker(T) = \operatorname{span}\left\{\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}\right\}$$

For range:

y_1, y_2 can be anything

$$\Rightarrow \operatorname{ran}(T) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}.$$

Theorems

$T: V \to W$ is a linear transformation

1. $\dim(\ker(T)) + \dim(\operatorname{ran}(T)) = V$
2. $\dim(RS(A)) = \dim(CS(A))$