

Section 1.2: Matrices and Matrix Operations.

Example:

$$x_1 + 2x_2 - 3x_3 + 4x_4 = 2$$

$$2x_1 - 4x_2 + 6x_3 - 5x_4 = 10$$

$$x_1 - 6x_2 + 9x_3 - 9x_4 = 8$$

$$3x_1 - 2x_2 + 4x_3 - x_4 = 12$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -3 & 4 & 2 \\ 2 & -4 & 6 & -5 & 10 \\ 1 & -6 & 9 & -9 & 8 \\ 3 & -2 & 4 & -1 & 12 \end{bmatrix} \begin{array}{l} \\ -2R_1 \\ -R_1 \\ -3R_1 \end{array} \Rightarrow \begin{bmatrix} 1 & 2 & -3 & 4 & 2 \\ 0 & -8 & 12 & -13 & 6 \\ 0 & -8 & 12 & -13 & 6 \\ 0 & -8 & 13 & -13 & 6 \end{bmatrix} \begin{array}{l} \\ \\ -R_2 \\ -R_2 \end{array}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -3 & 4 & 2 \\ 0 & -8 & 12 & -13 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{array}{l} \\ \\ \updownarrow \\ \end{array} \Rightarrow \begin{bmatrix} 1 & 2 & -3 & 4 & 2 \\ 0 & -8 & 12 & -13 & 6 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_3 = 0$$

$$-8x_2 - 13x_3 = 6 \Rightarrow x_2 = -\frac{13}{8}x_3 - \frac{3}{4}$$

$$x_1 + 2x_2 + 4x_3 = 2$$

$$\Rightarrow x_1 - \frac{13}{4}x_3 - \frac{3}{2} + 4x_3 = 2$$

$$\Rightarrow x_1 = -\frac{3}{4}x_3 + \frac{7}{2}$$

Matrix Algebra

Matrix = collection of numbers enclosed in a rectangle.

2×2 : $\begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}$, 1×1 : $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, 1×5 : $[4 \ 5 \ 6 \ 7 \ 8]$.

rows columns

3×4 : $\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{34} & a_{35} \end{bmatrix} = A$

\downarrow
 $A = [a_{ij}]$
 $\text{ent}_{23}(A) = a_{23}$

$B = \begin{bmatrix} -1 & 2 & 1 \\ 5 & 4 & -9 \\ 3 & -4 & 7 \end{bmatrix}$ square 3×3 matrix

Diagonal Entries.

$X = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$ vector, or column vector.

Addition:

$A + B = [a_{ij} + b_{ij}]$ ← Only works for matrices of same size.

$cA = [ca_{ij}]$

Real Number, scalar.

Example:

$$A = \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 \\ 4 & 3 \end{bmatrix}$$

$$\begin{aligned} \Rightarrow 2A + B &= 2 \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -2 \\ 4 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 6 \\ 10 & 2 \end{bmatrix} + \begin{bmatrix} 1 & -2 \\ 4 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 5 & 4 \\ 14 & 5 \end{bmatrix} \end{aligned}$$

Matrix Multiplication.

$$AB = [p_{ij}]$$

$$\begin{aligned} p_{ij} &= a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj} \\ &= \sum_{k=1}^n a_{ik}b_{kj} \end{aligned}$$

Example:

$$\begin{aligned} 1. \begin{bmatrix} 2 & -1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 3 & -1 \end{bmatrix} &= \begin{bmatrix} 2 \cdot 0 - 1 \cdot 3 & 2 \cdot 1 + (-1) \cdot (-1) \\ 1 \cdot 0 + 5 \cdot 3 & 1 \cdot 1 + 5 \cdot (-1) \end{bmatrix} \\ &= \begin{bmatrix} -3 & 3 \\ 15 & -4 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} 2. \begin{bmatrix} 1 & -1 & 4 \\ 2 & -3 & 6 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 2 & -1 \end{bmatrix} &= \begin{bmatrix} 6 & -2 \\ 5 & 1 \\ 3 & 1 \end{bmatrix} \\ \underbrace{\hspace{10em}}_{3 \times 3} \quad \underbrace{\hspace{10em}}_{3 \times 2} & \quad \underbrace{\hspace{10em}}_{3 \times 2} \end{aligned}$$

$$3. \begin{bmatrix} 1 & -1 & 4 \\ 2 & -3 & 6 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 15 \\ 24 \\ 4 \end{bmatrix}.$$

Return to Equations.

Write in matrix form

$$2x - y + 4z = 1$$

$$x + y - z = 4$$

$$y + 3z = 5$$

$$x + y = 2$$

$$\Rightarrow \begin{bmatrix} 2 & -1 & 4 \\ 1 & 1 & -1 \\ 0 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 5 \\ 2 \end{bmatrix}$$

4×3 3×1 4×1

Example:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 1 \\ 0 & -1 \\ 2 & 7 \end{bmatrix}$$

2×3 3×2

$$\Rightarrow AB = \begin{bmatrix} 12 & 27 \\ 8 & -4 \end{bmatrix}$$

Example:

Suppose A, B, C are matrices with the following sizes

$$\frac{A}{3 \times 4}, \quad \frac{B}{4 \times 7}, \quad \frac{C}{7 \times 3}$$

AB is defined and is 3×7	BC is defined and is 4×3
AC is not defined	CA is defined and is 7×4
BA is not defined	CB is not defined.

Linear Combinations

$$\begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

3×3 3×1

→

$$2 \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 4 \end{bmatrix} + \begin{bmatrix} -3 \\ -2 \\ -1 \end{bmatrix} + \begin{bmatrix} 6 \\ -9 \\ -6 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$