

## Lecture 21: Diagonalization and Jordan Canonical Form.

Suppose  $A \in \mathbb{R}^{n \times n}$  with linearly independent eigenvectors with eigenvalues  $\lambda_1, \dots, \lambda_n$ .

$$V = [\vec{v}_1 \mid \dots \mid \vec{v}_n]$$

$$\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \dots & \\ 0 & & \lambda_n \end{bmatrix}$$

$$A\vec{v}_i = \lambda_i \vec{v}_i = \vec{v}_i (\lambda_i)$$

$$\Rightarrow A \cdot V = V \Lambda$$

$\Downarrow$

$$A = V \Lambda V^{-1} \quad \text{and} \quad \Lambda = V^{-1} A V$$

$\uparrow$   
standard representation

$\uparrow [A]_{\mathcal{B}}$   
 $\downarrow$   
representation in basis of eigenvectors.

A is diagonalizable if and only if

$$\dim(E_{\lambda_1}) + \dots + \dim(E_{\lambda_n}) = n.$$

Example:

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}$$

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda - 1 & 3 \\ 2 & \lambda - 2 \end{bmatrix}$$

From last time:

$$\lambda_1 = 4, \lambda_2 = -1$$
$$\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 3/2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} -1 & 3/2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 3/2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$$

$\uparrow$   
P

$\uparrow$   
P<sup>-1</sup>

$$\left[ \begin{array}{cc|cc} -1 & \frac{3}{2} & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array} \right] \xrightarrow{+R1} \left[ \begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ -1 & \frac{3}{2} & 1 & 0 \end{array} \right] \xrightarrow{+R1} \left[ \begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & \frac{5}{2} & 1 & 1 \end{array} \right] \xrightarrow{-\frac{1}{2}R2}$$

$$\Rightarrow \left[ \begin{array}{cc|cc} 1 & 0 & -\frac{5}{2} & -\frac{3}{2} \\ 0 & \frac{5}{2} & 1 & 1 \end{array} \right] \Rightarrow \left[ \begin{array}{cc|cc} 1 & 0 & -\frac{5}{2} & -\frac{3}{2} \\ 0 & 1 & \frac{2}{5} & \frac{2}{5} \end{array} \right]$$

$$\begin{bmatrix} -\frac{5}{2} & -\frac{3}{2} \\ \frac{2}{5} & \frac{2}{5} \end{bmatrix} \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} -1 & \frac{3}{2} \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$$

### Jordan Canonical Form

If  $B = P^{-1}AP$  we say  $A$  and  $B$  are similar.

→ If  $A$  has  $n$  linearly ind. eigenvectors it is similar to a diagonal matrix.

→ If  $A$  has less than  $n$  linearly ind. eigenvectors  $A$  is similar to Jordan matrix.

### Example:

$\det(\lambda I - A) = (\lambda - 4)(\lambda - 2)^3$  what are the possible Jordan canonical forms?

$$A \sim \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \text{ or } A \sim \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$