Section 1.3: Inverses of Matrices

Example:

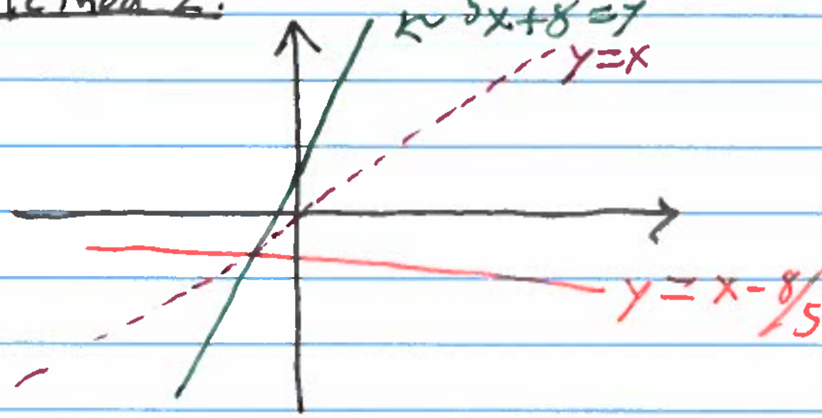
If $f(x) = 5x+8$, what is $f^{-1}(x)$?

- Method 1:

  $f(x) =$ five times $x + 8 \quad = 5x+8$

  $f^{-1}(x) = x - 8$ divided by $5 = \frac{x-8}{5}$.

- Method 2:



(reflection across line $y=x$)

- Method 3:

  We know:

  $$f(f^{-1}(x)) = x$$

  $$\Rightarrow 5f^{-1}(x) + 8 = x$$

  $$\Rightarrow f^{-1}(x) = \frac{x-8}{5}.$$

  → Just solve for the inverse!!

Definition - If A, B are square matrices so that

$$AB = BA = I,$$

then A is invertible and $A^{-1} = B$. If no such B exists then A is called singular.

Example:

- If $x, y \in \mathbb{R}$ and $xy = 0$ then $x = 0$ or $y = 0$.
- If $A, B \in \mathbb{R}^{n \times n}$ does $AB = 0$ imply $A = 0$ or $B = 0$??
- If $A, B \in \mathbb{R}^{n \times n}$ and A is invertible then $AB = 0$ implies $B = 0$.

Example:

$B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$, $A = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$ are inverses

$$BA = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example:

The matrix $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ is singular.

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{11} & b_{12} \end{bmatrix} = I \Rightarrow b_{11} = 1 \text{ and } b_{11} = 0.$$

(Contradiction)

Theorem - Inverses are unique

proof:

Suppose $AB=I$, $BA=I$, $AC=I$, $CA=I$.

$$\Rightarrow AB=I$$
$$\Rightarrow CAB=CI$$
$$\Rightarrow I\cdot B=C\cdot I$$
$$\Rightarrow B=C.$$

Definition - If A is invertible we define the inverse of A by A^{-1}.

Example:

Find the inverse of

$$A=\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

$$A\cdot B=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} b_{11}+2b_{21}+3b_{31} & b_{12}+2b_{22}+3b_{32} & b_{13}+2b_{23}+3b_{33} \\ 2b_{11}+5_{21}+3b_{31} & 2b_{12}+5b_{22}+3b_{32} & 2b_{13}+5b_{23}+3b_{33} \\ b_{11}+8b_{31} & b_{12}+8b_{32} & b_{13}+8b_{33} \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{array}{c|c|c} b_{11}+2b_{12}+3b_{31}=1 & b_{12}+2b_{22}+3b_{23}=0 & b_{13}+2b_{23}+3b_{33}=0 \\ 2b_{11}+5b_{21}+3b_{31}=0 & 2b_{12}+5b_{22}+3b_{23}=1 & 2b_{13}+5b_{23}+3b_{33}=0 \\ b_{11}+8b_{31}=0 & b_{12}+8b_{23}=0 & b_{13}+8b_{33}=1. \end{array}$$

$$\Downarrow$$

Augmented Matrix.

$$\begin{array}{l}\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]\end{array}\begin{array}{l} \\ -2R1 \\ -R1 \end{array} \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right]\begin{array}{l} -2R2 \\ \\ +2R2 \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 9 & 5 & -1 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right]\begin{array}{l} \\ \\ \times(-1) \end{array} \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 9 & 5 & -1 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]\begin{array}{l} -9R3 \\ +3R3 \\ \\ \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

$b_{11}=-40,\ b_{12}=16,\ b_{13}=9$
$b_{21}=13,\ b_{22}=-5,\ b_{23}=-3$
$b_{31}=5,\ b_{32}=-2,\ b_{33}=-1.$

$$\Rightarrow A^{-1}=\begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

Example:

Solve the following system of equations

$$\begin{aligned} 2x+6y+6z&=a \\ 2x+7y+6z&=b \\ 2x+7y+7z&=c \end{aligned} \Rightarrow A\vec{x}=\begin{bmatrix} a \\ b \\ c \end{bmatrix} \Rightarrow \vec{x}=A^{-1}\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Augmented matrix:

$$\begin{array}{cl} \left[\begin{array}{ccc:ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 2 & 7 & 6 & 0 & 1 & 0 \\ 2 & 7 & 7 & 0 & 0 & 1 \end{array}\right] & \begin{array}{l} \div 2 \\ \\ \\ \end{array} \end{array} \rightarrow \begin{array}{cl} \left[\begin{array}{ccc:ccc} 1 & 3 & 3 & \frac{1}{2} & 0 & 0 \\ 2 & 7 & 6 & 0 & 1 & 0 \\ 2 & 7 & 7 & 0 & 0 & 1 \end{array}\right] & \begin{array}{l} \\ -2R1 \\ -2R1 \end{array} \end{array}$$

$$\Rightarrow \begin{array}{cl} \left[\begin{array}{ccc:ccc} 1 & 3 & 3 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right] & \begin{array}{l} -3R2 \\ \\ -R2 \end{array} \end{array} \rightarrow \begin{array}{cl} \left[\begin{array}{ccc:ccc} 1 & 0 & 3 & \frac{7}{2} & -3 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] & \begin{array}{l} -3R3 \\ \\ \\ \end{array} \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & \frac{7}{2} & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \Rightarrow A^{-1}=\begin{bmatrix} \frac{7}{2} & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Therefore,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{7}{2} & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \frac{7}{2}a \\ -a+b \\ -b+c \end{bmatrix}$$

$$\boxed{\begin{aligned} x&=\tfrac{7}{2}a \\ y&=b-a \\ z&=c-b \end{aligned}}$$

Example:

If $A = \begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix}$, find A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 2 & 4 & -1 & 0 & 1 & 0 \\ -1 & 2 & 5 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ -2R1 \\ +R1 \end{array} \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 8 & 9 & 1 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \\ +R2 \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \end{array}\right] \Rightarrow \begin{array}{l} -1 = 0? \\ 1 = 0? \end{array} \Rightarrow \text{No inverse!}$$

Theorem - The following statements are equivalent.

(a) A is invertible

(b) $A\vec{x} = 0$ has only the trivial solution.

(c) Reduced row echelon form of A is I.

(d) $A\vec{x} = \vec{b}$ has only one solution for all $\vec{b}$.

proof:

If A is invertible then it can be row reduced to the identity.

Theorem - If A, B are invertible then

$$(AB)^{-1} = B^{-1}A^{-1}$$

proof:

$$(A \cdot B) \cdot (B^{-1} A^{-1}) = AB \cdot B^{-1} A^{-1} = A \cdot I \cdot A^{-1} = A \cdot A^{-1} = I.$$