Section 1.5: Determinants.

Big Picture:

Solve

$$A\vec{x}=\vec{b}, \quad \vec{x}=\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad \vec{b}=\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}.$$

1. Row reduce (Can take forever).
2. Invert A (Equivalent to row reducing)

$$\vec{x}=A^{-1}\vec{b}.$$

3. How do we know when an inverse exists??
4. Inverse does not exist when ...
   i) $A\vec{x}=0$ has nontrivial rows
   ii) A row reduces to row with all zeros.

How else can we figure out when an inverse exists:

$$A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A=\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right]\begin{array}{l} /a \\ /c \end{array} \Rightarrow \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 1 & d/c & 0 & 1/c \end{array}\right]\begin{array}{l} \\ -R1 \end{array}$$

$$\Rightarrow A=\left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & b/a-d/c & -1/a & 1/c \end{array}\right]$$

$\Rightarrow$ Inverse exists when

$$\frac{b}{a}-\frac{d}{c}\neq 0$$

$$\Rightarrow ad-bc\neq 0.$$

$$\Rightarrow \det(A_{2\times 2})=ad-bc \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Determinants 3×3:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

M_{ij} = matrix formed by deleting i-row, j-th column (Minor).

$$M_{21} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \cancel{a_{21}} & \cancel{a_{22}} & \cancel{a_{23}} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix}$$

$$\det(A) = a_{11} \det(M_{11}) - a_{12} \det(M_{12}) + a_{13} \det(M_{13})$$

$\downarrow$ compute using formula for 2×2 matrices.

Example:

$$A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$$

$$\det(A) = 3 \cdot \det\begin{pmatrix} 5 & 6 \\ 4 & 8 \end{pmatrix} - 1 \cdot \det\begin{pmatrix} 2 & 6 \\ 1 & 8 \end{pmatrix} + 4 \det\begin{pmatrix} 2 & 5 \\ 1 & 4 \end{pmatrix}$$

$$= 3 \cdot (5 \cdot 8 - 24) - 1(16 - 6) - 4(8 - 5)$$

$$= 3 \cdot 16 - 10 - 4 \cdot 3$$

$$= 48 - 10 - 12$$

$$= 26.$$

Cofactors are defined by:

$$C_{ij} = (-1)^{i+j} \det(M_{ij})$$

$$\begin{array}{c} (-1)^2 \quad (-1)^3 \\ (-1)^3 \end{array} \begin{pmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & & & \vdots & \end{pmatrix}$$

and

$$\det(A) = \sum_{i=1}^{n} C_{ij} \det(M_{ij}) \rightarrow \text{Sum over rows}$$

$$\det(A) = \sum_{j=1}^{n} C_{ij} \det(M_{ij}) \rightarrow \text{Sum over columns.}$$

Example:

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix} \qquad \begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

$$\det(A) = 3\cdot(8-12) - 1\cdot(-11) = 3(-4) + 11 = -1 \text{ (first row)}$$

$$\det(A) = 0\cdot(12) - 3\cdot(12-5) - 2(3\cdot(-4)+2) = -1 \text{ (third column)}$$

Theorem - If A is triangular the $\det(A) = a_{11} a_{22} \cdots a_{nn}$.

proof:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{21} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \Rightarrow \det(A) = a_{11} \det \begin{pmatrix} a_{21} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{nn} \end{pmatrix}$$

Continue by induction.

Theorem - $\det(A^T) = \det(A)$

Theorem:

1. If B is a matrix obtained by switching two rows of A then $\det(A) = -\det(B)$.
2. If B is a matrix obtained from A by multiplying a row of A by a scalar c, then $\det(B) = c\det(A)$
3. If B is a matrix obtained from A by replacing a row of A by itself plus a multiple of another row of A, then $\det(B) = \det(A)$.

1. $$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = B$$

$$\det(A) = -\det(B)$$

2. $$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \times c \Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = B$$

$$\det(A) = c\det(B)$$

3. $$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} -2R1 \Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21}-2a_{11} & a_{22}-2a_{12} & a_{23}-2a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\det(A) = \det(B).$$

Examples

1. $A=\begin{bmatrix}0&1&5\\3&-6&9\\2&6&1\end{bmatrix}$, $A=\begin{bmatrix}2&1&3&1\\1&0&1&1\\0&2&1&0\\0&1&2&3\end{bmatrix}$

$$\det\left(\begin{bmatrix}0&1&5\\3&-6&9\\2&6&1\end{bmatrix}\right)=-\det\left(\begin{bmatrix}3&-6&9\\0&1&5\\2&6&1\end{bmatrix}\right)=-3\det\left(\begin{bmatrix}1&-2&3\\0&1&5\\2&6&1\end{bmatrix}\right)\begin{array}{l}\\ \\ -2R1\end{array}$$

$$=-3\det\left(\begin{bmatrix}1&-2&3\\0&1&5\\0&10&-5\end{bmatrix}\right)\begin{array}{l}\\ \\ -10R2\end{array}=-3\det\left(\begin{bmatrix}1&-2&3\\0&1&5\\0&0&-55\end{bmatrix}\right)$$

$$\Rightarrow \det(A)=165.$$

2. $$\det\left(\begin{bmatrix}2&1&3&1\\1&0&1&1\\0&2&1&0\\0&1&2&3\end{bmatrix}\right)=-\det\left(\begin{bmatrix}1&0&1&1\\2&1&3&1\\0&2&1&0\\0&1&2&3\end{bmatrix}\right)\begin{array}{l}\\ -2R2\\ \\ \\ \end{array}$$

$$=-\det\left(\begin{bmatrix}1&0&1&1\\0&1&3&1\\0&2&1&0\\0&1&2&3\end{bmatrix}\right)=-\det\left(\begin{bmatrix}1&3&1\\2&1&0\\1&2&3\end{bmatrix}\right)\begin{array}{l}\\ -2R1\\ -R1\end{array}$$

$$=-\det\left(\begin{bmatrix}1&3&1\\0&-5&-2\\0&-1&2\end{bmatrix}\right)=-\det\left(\begin{bmatrix}-5&-2\\-1&2\end{bmatrix}\right)=-(-10-2)=12.$$