Section 2.2: Subspaces and Spanning Sets.

Definition - A subset W of V is a subspace of V if W is also a vector space under scalar multiplication and addition in V.

Example:

Let W be vectors of the form

$$\begin{bmatrix} x \\ x \end{bmatrix},$$

$W = \{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : y = x \}$. $V = \mathbb{R}^2 = \{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \}$.

W is a subspace.

proof:

1. Let $\vec{v}_1, \vec{v}_2 \in W$. Therefore, there exists $v_1, v_2 \in \mathbb{R}$ such that

$$\vec{v}_1 = \begin{bmatrix} v_1 \\ v_1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} v_2 \\ v_2 \end{bmatrix}$$

$$\Rightarrow \vec{v}_1 + \vec{v}_2 = \begin{bmatrix} v_1 + v_2 \\ v_1 + v_2 \end{bmatrix} \in W$$

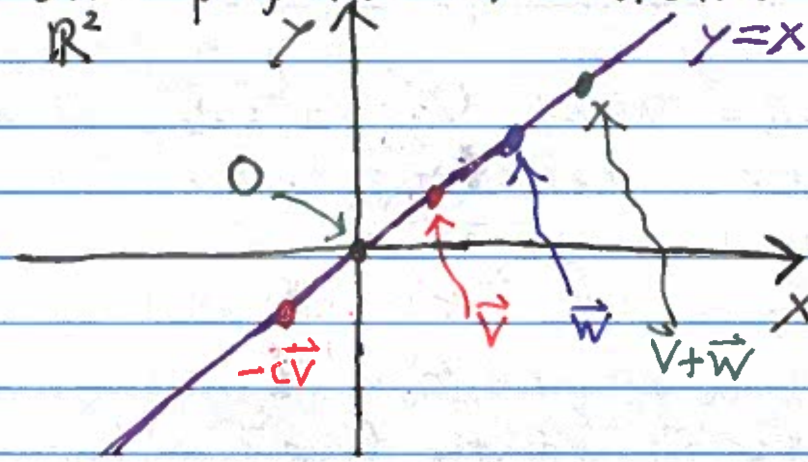
closure.

2. $c\vec{v}_1 = \begin{bmatrix} c\vec{v}_1 \\ c\vec{v}_1 \end{bmatrix} \in W$

3. $0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in W$

4. $-\vec{v} = \begin{bmatrix} -v_1 \\ -v_1 \end{bmatrix} \in W.$

All other properties are satisfied by default.



<u>Theorem</u>- Let W be a subset of a vector space V.
W is a subspace of V if and only if:
(a) $\vec{u}, \vec{v} \in W \Rightarrow \vec{u}+\vec{v} \in W$
(b) $\vec{u} \in \vec{W} \Rightarrow c\vec{u} \in W$

<u>Examples:</u>

1. $W = \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$ is not a subspace of $\mathbb{R}^2$.
   ~~proof~~:
   $(1,1) \in W \Rightarrow -1(1,1) = (-1,-1) \notin W$.

2. $W = \{n \times n$ symetric matrices$\}$ is a subspace of $\mathbb{R}^{n \times n}$.
   ~~proof~~:
   - Sum of two symmetric matrices is symmetric
   - Scalar times symmetric matrix is symmetric.

Theorem - If $A\vec{x}=0$, the set of homogeneous solutions is a subspace of $\mathbb{R}^n$. (Nullspace, kernel)

proof:

a.) Suppose $\vec{x}_1, \vec{x}_2$ satisfy $A\vec{x}_1=0$, $A\vec{x}_2=0$. Therefore,

$$\begin{aligned} A(\vec{x}_1+\vec{x}_2) &= A\vec{x}_1 + A\vec{x}_2 \\ &= 0+0 \\ &= 0. \end{aligned}$$

b.) Suppose $\vec{x}_1$ satisfies $A\vec{x}_1=0$. Therefore,

$$\begin{aligned} A(c\vec{x}_1) &= c\,A\vec{x}_1 \\ &= c\cdot 0 \\ &= 0. \end{aligned}$$

Example:

Describe the nullspace of the following matrices

a.) $A=\begin{bmatrix} 1 & -2 & 3 \\ 2 & -4 & 6 \\ 3 & -6 & 9 \end{bmatrix}\begin{matrix} \\ -2R1 \\ -3R1 \end{matrix} \Rightarrow \begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$z=$ anything

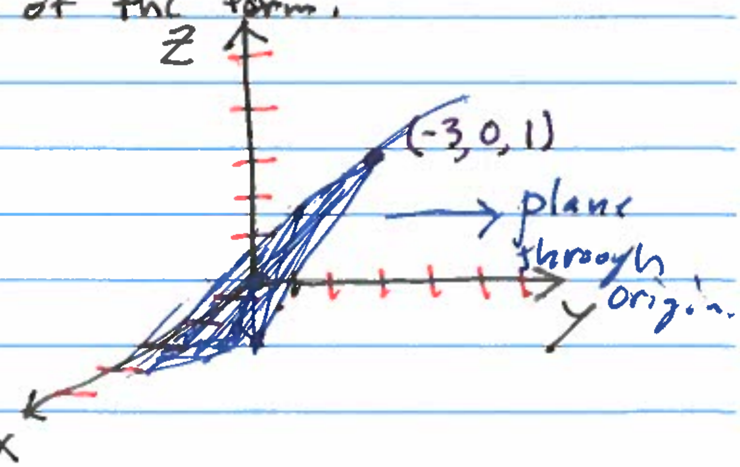
$y=$ anything

$x-2y+3z=0$

$\Rightarrow x=+2y-3z$

$NS(A)=$ set of vectors of the form:

$$\begin{bmatrix} 2y-3z \\ y \\ z \end{bmatrix}$$

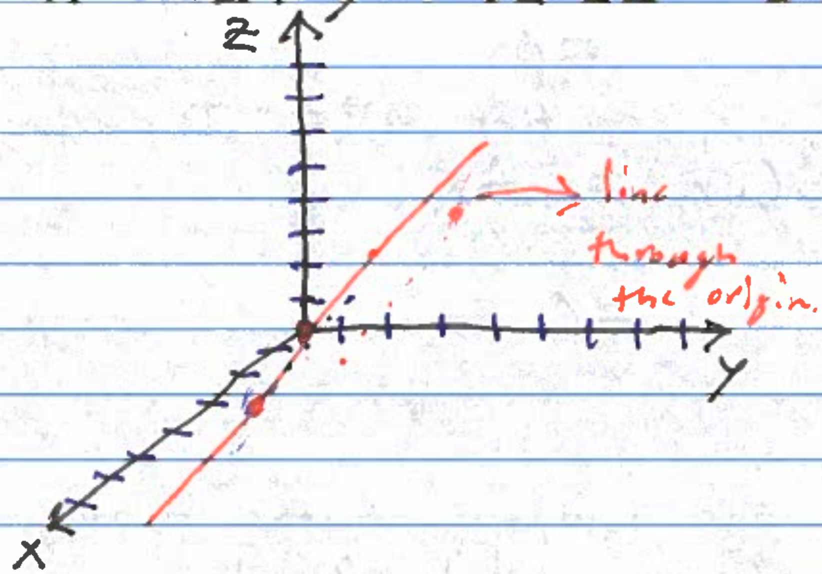


b.) $\begin{bmatrix}1 & -2 & 3\\ -3 & 7 & -8\\ -2 & 4 & -6\end{bmatrix}\begin{matrix}\\ +3R1\\ +2R1\end{matrix} \Rightarrow \begin{bmatrix}1 & -2 & 3\\ 0 & 1 & 1\\ 0 & 0 & 0\end{bmatrix}$

$\Rightarrow z = \text{arbitrary}$

$y = -z$

$x = -3z + 2y = -3z - 2z = -5z.$



c.) $\begin{bmatrix}1 & -2 & 3\\ -3 & 7 & -8\\ 4 & 1 & 2\end{bmatrix}\begin{matrix}\\ +3R1\\ -4R1\end{matrix} \Rightarrow \begin{bmatrix}1 & -2 & 3\\ 0 & 1 & 1\\ 0 & 9 & -10\end{bmatrix}\begin{matrix}\\ \\ -9R1\end{matrix} \Rightarrow \begin{bmatrix}1 & -2 & 3\\ 0 & 1 & 1\\ 0 & 0 & -19\end{bmatrix}$

$NA(A) = \{0\}.$

D.) $\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \Rightarrow NA(A) = \mathbb{R}^3.$

Definition – A vector $\vec{w}$ is a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ if it can be written in the form

$$\vec{w} = k_1\vec{v}_1 + \ldots + k_n\vec{v}_n,$$

where $k_1, \ldots, k_n \in \mathbb{R}$.

Example:

1. Express $\begin{bmatrix}1\\2\\3\end{bmatrix}$ as a linear combination of $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$

$$\begin{bmatrix}1\\2\\3\end{bmatrix} = k_1\begin{bmatrix}1\\0\\0\end{bmatrix} + k_2\begin{bmatrix}0\\1\\0\end{bmatrix} + k_3\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$\Rightarrow k_1 = 1,\ k_2 = 2,\ k_3 = 3$$

$$\Rightarrow \begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\\0\end{bmatrix} + 3\begin{bmatrix}0\\0\\1\end{bmatrix}.$$

2. Let $\vec{u} = \begin{bmatrix}1\\2\\-1\end{bmatrix}$, $\vec{v} = \begin{bmatrix}6\\4\\2\end{bmatrix}$.

a.) Is $\vec{w} = \begin{bmatrix}9\\2\\7\end{bmatrix}$ a linear combination of $\vec{u}$ and $\vec{v}$

b.) Is $\vec{w} = \begin{bmatrix}4\\-1\\8\end{bmatrix}$ is not a linear combination of $\vec{u}$ and $\vec{v}$.

$$\begin{bmatrix}9\\2\\7\end{bmatrix} = k_1\begin{bmatrix}1\\2\\-1\end{bmatrix} + k_2\begin{bmatrix}6\\4\\2\end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{cc|c}1 & 6 & 9\\2 & 4 & 2\\-1 & 2 & 7\end{array}\right] \begin{array}{l}\\ -2R1\\ +R1\end{array} \Rightarrow \left[\begin{array}{cc|c}1 & 6 & 9\\0 & -8 & -16\\0 & 8 & 16\end{array}\right] \begin{array}{l}\\ \\ +R2\end{array} \Rightarrow \left[\begin{array}{cc|c}1 & 6 & 9\\0 & 1 & 2\\0 & 0 & 0\end{array}\right]$$

$$\rightarrow k_2 = 2$$
$$k_1 = -3.$$

$$\Rightarrow \vec{w} = -3\vec{u} + 2\vec{v}.$$

3. $$\begin{bmatrix}1 & 6 & 4\\2 & 4 & -1\\-1 & 2 & 8\end{bmatrix} \begin{array}{l}\\ -2R1\\ +R1\end{array} \Rightarrow \begin{bmatrix}1 & 6 & 4\\0 & -8 & -9\\0 & 8 & 12\end{bmatrix}$$

No solution $\Rightarrow$ Not a linear combination

<u>Theorem</u> - If V is a vector space and $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are vectors in V, then the set of all linear combinations called the subspace of V spanned by $\vec{v}_1, \ldots, \vec{v}_n$ denoted by

$$\text{span}\{\vec{v}_1, \ldots, \vec{v}_n\}$$

is a subspace of V.

<u>proof</u>:

If $\vec{v}, \vec{w} \in V$ then there exists $c_1, \ldots, c_n, d_1, \ldots, d_n$ so that

$$\vec{v} = c_1\vec{v}_1 + \ldots + c_n\vec{v}_n$$
$$\vec{w} = d_1\vec{v}_1 + \ldots + d_n\vec{v}_n$$
$$\Rightarrow \vec{v} + \vec{w} = (c_1 + d_1)\vec{v}_1 + \ldots + (c_n + d_n)\vec{v}_n \in \text{span}\{\vec{v}_1, \ldots, \vec{v}_n\}.$$
$$k\vec{v} = k(c_1\vec{v}_1 + \ldots + c_n\vec{v}_n)$$
$$= kc_1\vec{v}_1 + \ldots + kc_n\vec{v}_n \in \text{span}\{\vec{v}_1, \ldots, \vec{v}_n\}.$$

<u>Example</u>:

1. Is $2x^2 + x + 1$ in $\text{span}\{x^2 + x, x^2 - 1, x + 1\}$?

$$a(x^2 + x) + b(x^2 - 1) + c(x + 1) = 2x^2 + x + 1$$
$$(a + b)x^2 + (a + c)x + (c - b) = 2x^2 + x + 1$$
$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$
$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 \end{array}\right] \begin{array}{c} \\ -R1 \\ \\ \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & -1 & 1 & -1 \\ 0 & -1 & 1 & 1 \end{array}\right] \begin{array}{c} \\ \\ -R2 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & -2 \end{array}\right]$$

There is no solution and thus $2x^2 + x + 1$ is not in $\text{span}\{x^2 + x, x^2 - 1, x + 1\}$.

2. Do $\begin{bmatrix}1\\-2\end{bmatrix}, \begin{bmatrix}2\\-4\end{bmatrix}$ span $\mathbb{R}^2$. Let $\vec{v} = (v_1, v_2)$ if $\vec{v} \in \text{span}\left\{\begin{bmatrix}1\\-2\end{bmatrix}, \begin{bmatrix}2\\-4\end{bmatrix}\right\}$ then

$$c_1\begin{bmatrix}1\\-2\end{bmatrix} + c_2\begin{bmatrix}2\\-4\end{bmatrix} = \begin{bmatrix}v_1\\v_2\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}1 & 2\\-2 & -4\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}v_1\\v_2\end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{cc|c}1 & 2 & v_1\\-2 & -4 & v_2\end{array}\right] +2R1$$

$$\Rightarrow \left[\begin{array}{cc|c}1 & 2 & v_1\\0 & 0 & v_2+2v_1\end{array}\right]$$

This does not have a solution for all $\vec{v}_1, \vec{v}_2$.

$\Rightarrow \left\{\begin{bmatrix}1\\-2\end{bmatrix}, \begin{bmatrix}2\\-4\end{bmatrix}\right\}$ does not span $\mathbb{R}^2$.