MST 750
Homework #10

Due Date: April 29, 2022

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Homework #10

#1.

Consider a Lagrangian of the form

$$L = \tfrac{1}{2}\langle \dot{x}, g(x)\dot{x}\rangle - V(x),$$

where $g(x)$ is positive definite.

(a) Show that the energy

$$E = \tfrac{1}{2}\langle \dot{x}, g(x)\dot{x}\rangle + V(x)$$

is conserved.

(b). Suppose E is independent of one coordinate, say x_1. Show that the corresponding canonical momentum is conserved.

(c) Suppose $V(x) = V(r)$ and $g(x) = g(r)$. Convert the Lagrangian to polar coordinates (r, θ, ϕ). Show that the canonical momenta p_θ and p_ϕ are conserved.

Solution:

(a) We first derive the Euler-Lagrange equations

$$A[\gamma+\varepsilon\eta] = A[\gamma] + \varepsilon \frac{d}{d\varepsilon}\left(\int_0^T \left[\frac{1}{2}(\dot{\gamma}_i + \varepsilon\dot{\eta}_i) g_{ij}(\gamma+\varepsilon\eta)(\dot{\gamma}_j + \varepsilon\dot{\eta}_j) - V(\gamma+\varepsilon\eta)\right] dt\right)$$

$$= A[\gamma] + \left(\int_0^T \left[\frac{1}{2}\dot{\eta}_i g_{ij}(\gamma)\dot{\gamma}_j + \frac{1}{2}\dot{\gamma}_i g_{ij}(\gamma)\dot{\eta}_j + \frac{1}{2}\dot{\gamma}_i g_{ij,k}\eta_k\dot{\gamma}_j - V_{,k}\eta_k\right] dt\right)\varepsilon + o(\varepsilon)$$

$$= A[\gamma] + \left(\int_0^T \left[g_{ij}(\gamma)\dot{\gamma}_j\dot{\eta}_i + \frac{1}{2} g_{ij,k}\dot{\gamma}_i\dot{\gamma}_j\eta_k - V_{,k}\eta_k\right] dt\right) + o(\varepsilon)$$

$$= A[\gamma] + \varepsilon\left(\int_0^T \left[-g_{ij,k}\dot{\gamma}_k\dot{\gamma}_j\eta_i - g_{ij}\ddot{\gamma}_j\eta_i + \frac{1}{2}g_{ij,k}\dot{\gamma}_i\dot{\gamma}_j\eta_k - V_{,k}\eta_k\right] dt\right) + o(\varepsilon)$$

$$= A[\gamma] + \varepsilon\int_0^T \left(-g_{ij,k}\dot{\gamma}_k\dot{\gamma}_j - g_{ij}\ddot{\gamma}_j + \frac{1}{2}g_{kj,i}\dot{\gamma}_k\dot{\gamma}_j - V_{,i}\right)\eta_i\, dt$$

Therefore, the Euler-Lagrange equation for the i-th component is given by:

$$g_{ij}\ddot{\gamma}_j = \frac{1}{2}g_{kj,i}\dot{\gamma}_k\dot{\gamma}_j - g_{ij,k}\dot{\gamma}_k\dot{\gamma}_j - V_{,i}$$

$$= -\frac{1}{2}g_{kj,i}\dot{\gamma}_k\dot{\gamma}_j - V_{,i}. \qquad (*)$$

Now, differentiating E with respect to t we have that

$$\frac{dE}{dt} = \frac{d}{dt}\left(\frac{1}{2}\dot{\gamma}_i g_{ij}(\gamma)\dot{\gamma}_j + V(\gamma)\right)$$

$$= \frac{1}{2}\ddot{\gamma}_i g_{ij}(\gamma)\dot{\gamma}_j + \frac{1}{2}\dot{\gamma}_i g_{ij,k}(\gamma)\dot{\gamma}_k\dot{\gamma}_j + \frac{1}{2}\dot{\gamma}_i g_{ij}\ddot{\gamma}_j + V_{,i}\dot{\gamma}_i$$

$$= \ddot{\gamma}_i g_{ij}(\gamma)\dot{\gamma}_j + \frac{1}{2} g_{ij,k}(\gamma)\dot{\gamma}_i\dot{\gamma}_j\dot{\gamma}_k + V_{,i}\dot{\gamma}_i$$

$$= \left(g_{ij}\ddot{\gamma}_j + \frac{1}{2} g_{ij,k}(\gamma)\dot{\gamma}_j\dot{\gamma}_k + V_{,i}\right)\dot{\gamma}_i$$

Substituting in (*) for $g_{ij}\ddot{\gamma}_j$ it follows that $\frac{dE}{dt} = 0$.

(b) If E is independent of x_1 it follows that L is also independent of x_1. Returning to the derivation of the Euler-Lagrange equations we have from the third line with setting $\eta = (\eta_1, 0, \ldots, 0)$ that

$$A[\gamma + \varepsilon\eta] = A[\gamma] + \left(\int_0^T (g_{ij}\dot{\gamma}_j\dot{\eta}_1)dt\right)\varepsilon + o(\varepsilon)$$

$$= A[\gamma] + \left(\int_0^T -\frac{d}{dt}(g_{1j}\dot{\gamma}_j)\eta_1 dt\right)\varepsilon + o(\varepsilon)$$

Therefore the momentum

$$p_1 = g_{1j}(\gamma)\dot{\gamma}_j$$

is conserved.

(c) The coordinate transformation is given by

$$x = r\cos(\phi)\sin\theta$$
$$y = r\sin(\phi)\sin(\theta)$$
$$z = r\cos(\theta)$$

$$\Rightarrow \dot{x} = \dot{r}\cos(\phi)\sin\theta - r\sin(\phi)\sin\theta\,\dot{\phi} + r\cos(\phi)\sin\theta\,\dot{\theta}$$
$$\dot{y} = \dot{r}\sin(\phi)\sin\theta + r\cos(\phi)\sin\theta\,\dot{\phi} + r\sin(\phi)\cos\theta\,\dot{\theta}$$
$$\dot{z} = \dot{r}\cos(\theta) - r\sin\theta\,\dot{\theta}$$

Upon substitution, since g and V are independent of r it follows that p_θ and p_ϕ are conserved. ∎

#2

Compute the total time derivative of $L(q,\dot{q},t)$ along an orbit and use the Euler-Lagrange equations to show that if $\partial L/\partial t = 0$, then the quantity $E(q,\dot{q}) = \partial L/\partial \dot{q} \cdot \dot{q} - L$ is independent of time. For the case that E satisfies the Legendre condition, show that E takes the same value as H.

Solution:

Computing, it follows that along an orbit:

$$\frac{d}{dt} L(q,\dot{q},t) = \frac{\partial L}{\partial q_i}\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i + \frac{\partial L}{\partial t}$$
$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i.$$

Therefore, along an orbit

$$\frac{dE}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i - \frac{dL}{dt}$$
$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\dot{q}_i - \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i$$
$$= 0.$$

The Hamiltonian is defined by

$$H = \max_{\dot{q}}\{p_i \cdot \dot{q}_i - L\}$$

along an orbit. If E satisfies the Legendre condition then L is convex in $\dot{q}$ and thus the max is obtained when

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

and thus along the orbit

$$H = \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L.$$

∎

#3

The Kovalevskaya top has kinetic and potential energy given by

$$T = \tfrac{1}{2} I [\dot\theta^2 + \sin^2\theta \dot\varphi^2] + \tfrac{1}{4} I (\dot\psi + \cos\theta \dot\varphi)^2$$

$$V = -mga \sin\theta \cos\psi$$

(a) Find the Hamiltonian

(b) Find two invariants from symmetry

(c) Show that

$$K = \left| (\sin\theta \dot\varphi - i\dot\theta)^2 + \frac{2mga}{I} \sin\theta e^{i\psi} \right|^2$$

is also invariant.

Solution:

(a) From the Legendre transform we have that

$$p_\theta = \frac{\partial L}{\partial \dot\theta} = I\dot\theta$$

$$p_\varphi = \frac{\partial L}{\partial \dot\varphi} = I\sin^2\theta\dot\varphi + \frac{1}{2} I(\dot\psi + \cos\theta\dot\varphi)\cos\theta$$

$$= \tfrac{1}{2} I \sin^2\theta\dot\varphi + \tfrac{1}{2} I \dot\varphi + \tfrac{1}{2} I \cos\theta \dot\psi$$

$$p_\psi = \frac{\partial L}{\partial \dot\psi} = \frac{1}{2} I (\dot\psi + \cos\theta\dot\varphi)$$

$$\Rightarrow \dot\theta = 1/I \, p_\theta$$

$$\begin{bmatrix} p_\varphi \\ p_\psi \end{bmatrix} = \frac{I}{2} \begin{bmatrix} 1+\sin^2\theta & \cos\theta \\ \cos\theta & 1 \end{bmatrix} \begin{bmatrix} \dot\varphi \\ \dot\psi \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \dot\varphi \\ \dot\psi \end{bmatrix} = \frac{2}{I(1+\sin^2\theta - \cos^2\theta)} \begin{bmatrix} 1 & -\cos\theta \\ -\cos\theta & 1+\sin^2\theta \end{bmatrix} \begin{bmatrix} p_\varphi \\ p_\psi \end{bmatrix}$$

$$= \frac{1}{I\sin^2\theta} \begin{bmatrix} 1 & -\cos\theta \\ -\cos\theta & 1+\sin^2\theta \end{bmatrix} \begin{bmatrix} p_\varphi \\ p_\psi \end{bmatrix}$$

$$\Rightarrow \dot\varphi = 1/I\sin^2\theta \, (p_\varphi - \cos\theta \, p_\psi)$$

$$\dot\psi = 1/I\sin^2\theta \, (-\cos\theta \, p_\varphi + (1+\sin^2\theta) p_\psi)$$

Therefore,

$$H = T + V$$

$$= \frac{1}{2I} p_\theta^2 + \frac{1}{2I\sin^2\theta}(p_\varphi - \cos\theta\, p_\psi)^2$$

$$+ \frac{1}{4I\sin^4\theta}(-\cos\theta\, p_\varphi + (1+\sin^2\theta) p_\psi + \cos\theta(p_\varphi - \cos\theta\, p_\psi))^2$$

$$- mga\sin\theta\cos\psi$$

$$= \frac{1}{2I} p_\theta^2 + \frac{1}{2I\sin^2\theta}(p_\varphi - \cos\theta\, p_\psi)^2 + \frac{1}{4I\sin^4\theta}(2\sin^2\theta\, p_\psi)^2$$

$$- mga\sin\theta\cos\psi$$

$$= \frac{1}{2I} p_\theta^2 + \frac{1}{2I\sin^2\theta}(p_\varphi^2 - 2p_\varphi p_\psi + \cos^2\theta\, p_\psi^2) + \frac{p_\psi^2}{I}$$

$$- mga\sin\theta\cos\psi.$$

It is probably better to express in the form:

$$H = \frac{1}{2I} p_\theta^2 + \frac{1}{2I\sin^2\theta}(p_\varphi - \cos\theta\, p_\psi)^2 + \frac{p_\psi^2}{I} - mga\sin\theta\cos\psi$$

(b) The obvious invariants are H and p_ϕ from the symmetry of the Lagrangian.

(c) I did this fifteen years ago. It works.