MST 750
Homework #1

Due Date: January 14, 2022

1. For the equation $\dot{x} = f(x)$, where f is continuously differentiable, show that if $x(t)$ is a solution then so is $x(t - t_0)$ for any t_0.

2. For the equation $\dot{x} = f(x)$, show that if $\cos(t)$ is a solution, then $-\sin(t)$ is also a solution.

3. For the equation $\ddot{x} = f(x)$, where f is continuously differentiable,

   (a) Show that if $1/(1+t)$ is a solution, then $1/(1-t)$ is also a solution.

   (b) Find f such that $1/(1+t)$ is a solution.

4. Solve the following differential equations:

   (a) $\dot{x} = x^3$

   (b) $\dot{x} = x(1-x)$

   (c) $\dot{x} = x(1-x) - c$

5. Suppose f, g are continuously differentiable functions defined on all of $\mathbb{R}$. Show that the equation

$$\dot{x} = f(x)g(t), \quad x(t_0) = x_0,$$

   locally has a unique solution if $f(x_0) \neq 0$. Give an implicit formula for the solution.

6. Solve the following differential equations:

   (a) $\dot{x} = \sin(t)x$

   (b) $\dot{x} = g(t)\tan(x)$

   (c) $\dot{x} = \sin(t)e^x$

7. Consider the following differential equation

$$\dot{x} = \begin{cases} -t\sqrt{|x|}, & x \geq 0, \\ t\sqrt{|x|}, & x \leq 0. \end{cases}$$

   (a) Show that if $x(t)$ is a solution then so is $-x(t)$ and $x(-t)$.

   (b) Show that without loss of generality, we can assume $x_0 \geq 0$ and $t \geq 0$.

   (c) Show that the initial value problem $x(0) = x_0$ has a unique global solution.

   (d) Show that global solutions can intersect. How does this not violate uniqueness of solutions?

Homework #1

#1.

For the equation $\dot{x}=f(x)$, where f is continuously differentiable, show that if $x(t)$ is a solution then so is $x(t-t_0)$ for any t_0.

Solution:

Let $y(t)=x(t-t_0)$. Then,

$$\dot{y}=\left.\frac{dx}{dt}\right|_{t-t_0}=f(x(t-t_0))=f(y).$$

∎

#2.

For the equation $\dot{x}=f(x)$, show that if $\cos(t)$ is a solution, then $-\sin(t)$ is also a solution.

Solution:

Since $-\sin(t)=\cos(t-3\pi/2)$ it follows from problem #1 that $-\sin(t)$ is also a solution.

∎

#3.

For the equation $\ddot{x}=f(x)$, where f is continuously differentiable,

(a) Show that if $1/1+t$ is a solution, then $1/1-t$ is also a solution.

(b) Find f such that $1/1+t$ is a solution.

Solution:

(a) Suppose $x(t)$ solves $\ddot{x}=f(x)$ and let $y(t)=x(-t)$. Therefore, $\ddot{y}=(-1)^2\ddot{x}(-t)=f(x(-t))=f(y)$. Therefore, if $1/1+t$ is a solution then so is $1/1-t$.

(b). If $x(t)=1/1+t$ then $\ddot{x}=2/(1+t)^3=2x(t)^3$. Therefore, $f(x)=2x^3$.

#5

Suppose f, g are continuously differentiable functions defined on all of $\mathbb{R}$. Show that

$$\dot{x} = f(x)g(t)$$
$$x(t_0) = x_0$$

locally has a unique solution if $f(x_0) \neq 0$. Give an implicit formula for the solution.

Solution:

Suppose $x(t)$ is a solution on the interval $[t_0-\eta, t_0+\eta]$. Therefore, since $f(x_0) \neq 0$ it follows from continuity of f and $x(t)$ that there exists $\delta < \eta$ such that $f(x(t))$ does not change sign on $[t_0-\delta, t_0+\delta]$. Consequently, for $t \in [t_0-\delta, t_0+\delta]$

$$\frac{1}{f(x)}\dot{x} = g(t)$$

$$\Rightarrow \int_{t_0}^{t} \frac{1}{f(x(t))}\frac{dx}{dt}\,dt = \int_{t_0}^{t} g(t)\,dt$$

$$\Rightarrow \int_{x_0}^{x} \frac{1}{f(x)}\,dx = \int_{t_0}^{t} g(t)\,dt.$$

Letting $F(x) = \int_{x_0}^{x} 1/f(x)\,dx$ and $G(t) = \int_{t_0}^{t} g(t)\,dt$ it follows that $F'(x) = 1/f(x)$ and thus is monotone. Therefore,

$$x(t) = F^{-1}(G(t))$$

is the unique solution on the interval $[t_0-\delta, t_0+\delta]$. ∎

#7

Consider the following differential equation

$$\dot{x} = \begin{cases} -t\sqrt{|x|}, & x \geq 0 \\ t\sqrt{|x|}, & x \leq 0. \end{cases}$$

(a). Show that if $x(t)$ is a solution then so is $-x(t)$ and $x(-t)$.

(b) Show that without loss of generality, we can assume $x_0 \geq 0$ and $t \geq 0$.

(c) Show that the initial value problem $x(0)=x_0$ has a unique global solution.

(d) Show that global solutions can intersect. How does this not violate uniqueness of solutions?

Solution:

(a). If $x(t)$ is a solution let $y(t)=-x(t)$ and $z(t)=x(-t)$. Therefore,

$$
\begin{aligned}
\dot{y} &= -\dot{x} \\
&= -\begin{cases} -t\sqrt{|x|}, & x\geq 0 \\ t\sqrt{|x|}, & x\leq 0 \end{cases} \\
&= \begin{cases} t\sqrt{|x|}, & x\geq 0 \\ -t\sqrt{|x|}, & x\leq 0 \end{cases} \\
&= \begin{cases} t\sqrt{|y|}, & -y\geq 0 \\ -t\sqrt{|y|} & -y\leq 0 \end{cases} \\
&= \begin{cases} t\sqrt{|y|}, & y\leq 0 \\ -t\sqrt{|y|}, & y\geq 0 \end{cases} \\
&= f(y,t),
\end{aligned}
$$

where

$$
f(x,t) = \begin{cases} -t\sqrt{|x|}, & x\geq 0 \\ t\sqrt{|x|}, & x\leq 0 \end{cases}.
$$

Moreover,

$$
\begin{aligned}
\dot{z} &= -\dot{x}(-t) \\
&= -f(x,-t) \\
&= -\begin{cases} t\sqrt{|x|}, & x\geq 0 \\ -t\sqrt{|x|}, & x\leq 0 \end{cases} \\
&= \begin{cases} -t\sqrt{|x|}, & x\geq 0 \\ t\sqrt{|x|}, & x\leq 0 \end{cases} \\
&= \begin{cases} -t\sqrt{|z|}, & z\geq 0 \\ t\sqrt{|z|}, & z\leq 0 \end{cases} \\
&= f(z,t).
\end{aligned}
$$

(b) By part (a) we can assume without loss of generality that $x_0 \geq 0$, and $t \geq 0$.

(c). Assuming $x_0 \geq 0$ and $t \geq 0$ we can assume

$$\dot{x} = -t\sqrt{x},$$
$$x(0) = x_0.$$

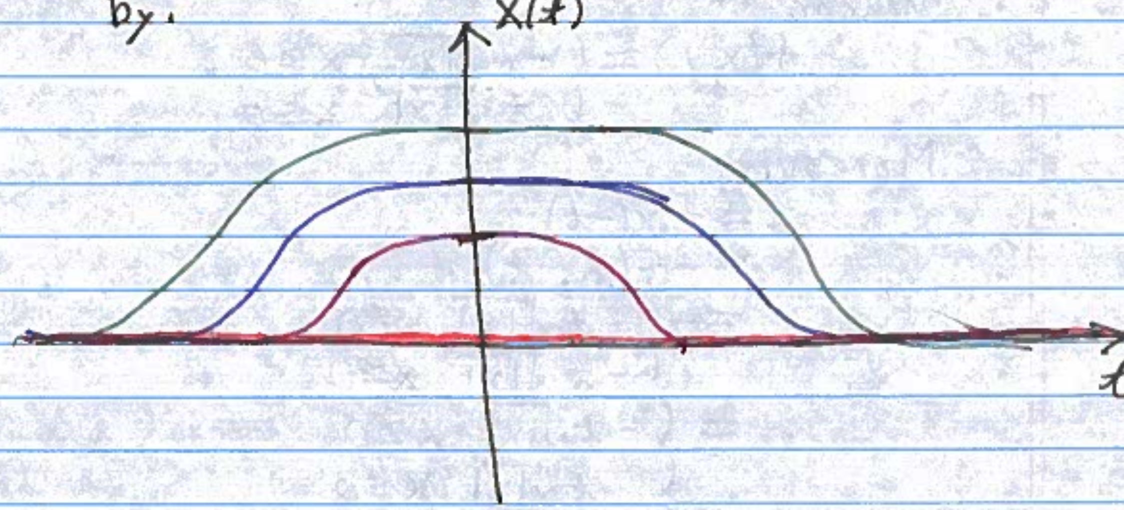
Therefore, for $x > 0$ we have

$$\int_{x_0}^{x} x^{-1/2}\,dx = \int_0^t -t\,dt$$
$$2x^{1/2} - 2x_0^{1/2} = \frac{-t^2}{2}$$
$$\Rightarrow x^{1/2} = x_0^{1/2} - \frac{t^2}{4}$$
$$\Rightarrow x = (x_0^{1/2} - t^2/4)^2.$$

However, if $x = 0$ we have $\dot{x} = 0$ and is thus constant. Consequently, the unique solution is given by:

$$x(t) = \begin{cases} (x_0^{1/2} - t^2/4)^2, & |t| \leq 2x_0^{1/4}, \\ 0 & ; |t| \geq 2x_0^{1/4} \end{cases}$$

(d) The solution curves for this problem are given by:



Solutions are unique if $x(t_0) \neq 0$