MST 750
Homework #2

Due Date: January 21, 2022

1. Consider the differential equation

$$\dot{x} = a(t)x + g(t),$$
$$x(t_0) = x_0.$$

   (a) Show by direct substitution that

$$x(t) = x_0 A(t, t_0) + \int_{t_0}^{t} A(t,s)g(s)ds,$$

   is a solution where

$$A(t, t_0) = \exp\left(\int_s^t a(s)ds\right).$$

   (b) Suppose $a \in \mathbb{R}$ is a constant and g is a continuous, nonnegative periodic function with period one, i.e. $g(t+1) = g(t)$. Find conditions on a and g such that $x(t)$ is a periodic solution.

2. For the following differential equations, sketch a phase portrait **and** determine the explicit form of any periodic orbits. Be sure to sketch the nullclines, the overall direction of the flow in each region separated by the nullclines, the location of any fixed points, and enough trajectories to illustrate the qualitative behavior of the flow. Feel free to use Mathematica to check your sketch. To determine the location of any periodic orbits, converting the system to polar coordinates $x = r\cos(\theta)$, $y = r\sin(\theta)$ could be useful.

   (a) $\begin{cases} \dot{x} &= y(y^2 - x^2) \\ \dot{y} &= -x(y^2 - x^2) \end{cases},$

   (b) $\begin{cases} \dot{x} &= -y + x(1 - x^2 - y^2) \\ \dot{y} &= x + y(1 - x^2 - y^2) \end{cases},$

   (c) $\begin{cases} \dot{x} &= x(10 - x^2 - y^2) \\ \dot{y} &= y(1 - x^2 - y^2) \end{cases},$

3. Consider the following system of differential equations:

$$\begin{cases} \dot{x} &= x + 3y^2 \\ \dot{y} &= -2x - y \end{cases}.$$

   (a) Calculate any fixed points and sketch a phase portrait for this system.

   (b) Show that this system is conservative, i.e. there exists a function $E(x, y)$ such that E is constant along solution trajectories. **Hint:** If E is is conserved then along solution curves we must have that

$$0 = \frac{d}{dt}E(x(t), y(t)) = \frac{\partial E}{\partial x}\frac{dx}{dt} + \frac{\partial E}{\partial y}\frac{dy}{dt}.$$

   (c) A homoclinic orbit is a solution that connects a fixed point to itself. Using the fact that E is constant, determine an explicit formula for any homoclinic orbits in this system.

4. Consider the following system of differential equations in polar coordinates:

$$\begin{cases} \dot{r} &= 0 \\ \dot{\theta} &= (r^2 - 1)(r^2\cos^2(\theta) + r\sin(\theta) + 1) \end{cases}.$$

(a) Sketch a phase portrait for this system.

(b) Determine if this system is conservative and find a conserved quantity.

(c) Show that the period of any periodic orbit is given by:

$$T(r_0) = \int_0^{2\pi} \frac{1}{(r_0^2 - 1)(r_0^2 \cos^2(\theta) + r_0 \sin(\theta) + 1)} d\theta,$$

where r_0 is the initial radial coordinate, i.e. $r(0) = r_0$.

(d) Show that in a neighborhood of the origin:

$$T(r_0) = 2\pi + ar_0^2 + o(r_0^2),$$

for some constant $a \neq 0$. Note, given a function $g : \mathbb{R} \mapsto \mathbb{R}$ such that $g(x)/x^k \to 0$ when $x \to 0$, we write $g(x) = o(x^k)$. **Hint:** Think about Taylor expanding or using the geometric series assuming $r_0 \ll 1$.

5. pg. 64, # 8.

6. pg. 64, # 10.