MST 750
Homework #2

Due Date: January 21, 2022

1. Consider the differential equation

$$\dot{x} = a(t)x + g(t),$$
$$x(t_0) = x_0.$$

   (a) Show by direct substitution that

$$x(t) = x_0 A(t, t_0) + \int_{t_0}^{t} A(t, s) g(s) ds,$$

   is a solution where

$$A(t, t_0) = \exp\left(\int_{s}^{t} a(s) ds\right).$$

   (b) Suppose $a \in \mathbb{R}$ is a constant and g is a continuous, nonnegative periodic function with period one, i.e. $g(t+1) = g(t)$. Find conditions on a and g such that $x(t)$ is a periodic solution.

2. For the following differential equations, sketch a phase portrait **and** determine the explicit form of any periodic orbits. Be sure to sketch the nullclines, the overall direction of the flow in each region separated by the nullclines, the location of any fixed points, and enough trajectories to illustrate the qualitative behavior of the flow. Feel free to use Mathematica to check your sketch. To determine the location of any periodic orbits, converting the system to polar coordinates $x = r\cos(\theta)$, $y = r\sin(\theta)$ could be useful.

   (a) $\begin{cases} \dot{x} &= y(y^2 - x^2) \\ \dot{y} &= -x(y^2 - x^2) \end{cases}$,

   (b) $\begin{cases} \dot{x} &= -y + x(1 - x^2 - y^2) \\ \dot{y} &= x + y(1 - x^2 - y^2) \end{cases}$,

   (c) $\begin{cases} \dot{x} &= x(10 - x^2 - y^2) \\ \dot{y} &= y(1 - x^2 - y^2) \end{cases}$,

3. Consider the following system of differential equations:

$$\begin{cases} \dot{x} &= x + 3y^2 \\ \dot{y} &= -2x - y \end{cases}.$$

   (a) Calculate any fixed points and sketch a phase portrait for this system.

   (b) Show that this system is conservative, i.e. there exists a function $E(x, y)$ such that E is constant along solution trajectories. **Hint:** If E is is conserved then along solution curves we must have that

$$0 = \frac{d}{dt} E(x(t), y(t)) = \frac{\partial E}{\partial x}\frac{dx}{dt} + \frac{\partial E}{\partial y}\frac{dy}{dt}.$$

   (c) A homoclinic orbit is a solution that connects a fixed point to itself. Using the fact that E is constant, determine an explicit formula for any homoclinic orbits in this system.

4. Consider the following system of differential equations in polar coordinates:

$$\begin{cases} \dot{r} &= 0 \\ \dot{\theta} &= (r^2 - 1)(r^2\cos^2(\theta) + r\sin(\theta) + 1) \end{cases}.$$

(a) Sketch a phase portrait for this system.

(b) Determine if this system is conservative and find a conserved quantity.

(c) Show that the period of any periodic orbit is given by:

$$T(r_0) = \int_0^{2\pi} \frac{1}{(r_0^2 - 1)(r_0^2 \cos^2(\theta) + r_0 \sin(\theta) + 1)} d\theta,$$

where r_0 is the initial radial coordinate, i.e. $r(0) = r_0$.

(d) Show that in a neighborhood of the origin:

$$T(r_0) = 2\pi + ar_0^2 + o(r_0^2),$$

for some constant $a \neq 0$. Note, given a function $g : \mathbb{R} \mapsto \mathbb{R}$ such that $g(x)/x^k \to 0$ when $x \to 0$, we write $g(x) = o(x^k)$. **Hint:** Think about Taylor expanding or using the geometric series assuming $r_0 \ll 1$.

5. pg. 64, # 8.

6. pg. 64, # 10.

Homework #2

#1

Consider

$$\dot{x} = a(t)x + g(t)$$
$$x(t_0) = x_0$$

(a) Show that

$$x(t) = x_0 A(t,t_0) + \int_{t_0}^{t} A(t,s)g(s)\,ds$$

is a solution where

$$A(t,s) = \exp\left(\int_s^t a(s)\,ds\right)$$

(b). Suppose $a \in \mathbb{R}$ is a constant and g is a continuous nonnegative periodic function with period one. Find conditions on a and g such that $x(t)$ is a periodic solution.

Solution:

(a) First, recall that

$$\frac{d}{dt}\int_s^{h(t)} f(t,s)\,ds = \frac{d}{dt}\Big(F(t,h(t)) - F(t,s)\Big)$$
$$= \left.\frac{\partial F}{\partial t}\right|_{(t,h)} + \left.\frac{\partial F}{\partial h}\right|_{(t,h)} \left.\frac{dh}{dt}\right|_t - \left.\frac{\partial F}{\partial t}\right|_{(t,h)}$$
$$= \frac{\partial}{\partial t}\int_s^h f(t,s)\,ds + f(t,h)\frac{dh}{dt}$$
$$= \int_s^h \frac{\partial f}{\partial t}\,ds + f(t,h)\frac{dh}{dt}.$$

Furthermore,

$$\frac{\partial A}{\partial t} = a(t)\exp\left(\int_s^t a(s)\,ds\right) = a(t)A(t,s).$$

Therefore,

$$\dot{x} = x_0 a(s)A(t,t_0) + \int_{t_0}^{t} \left.\frac{\partial A}{\partial t}\right|_{(t,s)} g(s)\,ds + A(t,t)g(t)$$

$$= x_0 a(s)A(t,t_0) + \int_{t_0}^{t} a(t)A(t,s)g(s)\,ds + g(t)$$
$$= a(s)\left(x_0 A(t,t_0) + \int_{t_0}^{t} A(t,s)g(s)\,ds\right) + g(t)$$
$$= a(s)\,x(t) + g(t).$$

Also,

$$x(t_0) = x_0 A(t_0, t_0) + \int_{t_0}^{t_0} A(t,s) g(s)\, ds$$
$$= x_0.$$

(b) If a is a constant it follows that

$$A(t,s) = \exp(a(t-s))$$

Therefore,

$$x(t) = x_0 \exp(a(t-t_0)) + \int_{t_0}^{t} \exp(a(t-s)) g(s)\, ds.$$
$$= x_0 e^{at} e^{-at_0} + e^{at} \int_{t_0}^{t} e^{-as} g(s)\, ds.$$

and thus

$$x(t_0+1) = x_0 e^{at_0} e^{-at_0} e^{a} + e^{at_0} e^{a} \int_{t_0}^{t_0+1} e^{-as} g(s)\, ds.$$

Equating $x(t_0+1) = x_0$, we have that

$$x_0 = x_0 e^{a} + e^{at_0} e^{a} \int_{t_0}^{t_0+1} e^{-as} g(s)\, ds.$$

Therefore,

$$x_0 (1 - e^{a}) = e^{at_0} e^{a} \int_{t_0}^{t_0+1} e^{-as} g(s)\, ds$$

$$\Rightarrow \frac{e^{a}}{1-e^{a}} \int_{t_0}^{t_0+1} e^{-a(s-t_0)} g(s)\, ds = x_0$$

$$\Rightarrow \frac{e^{a}}{1-e^{a}} \int_{0}^{1} e^{-as} g(s+t_0)\, ds = x_0.$$

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#2.

Sketch the following phase portraits and identify any closed orbis.

(a) $\begin{cases} \dot{x} = y(y^2 - x^2) \\ \dot{y} = -x(y^2 - x^2) \end{cases}$

(b). $\begin{cases} \dot{x} = -y + x(1 - x^2 - y^2) \\ \dot{y} = x + y(1 - x^2 - y^2) \end{cases}$

(c) $\begin{cases} \dot{x} = x(10 - x^2 - y^2) \\ \dot{y} = y(1 - x^2 - y^2). \end{cases}$

Solution:

(a) Note,

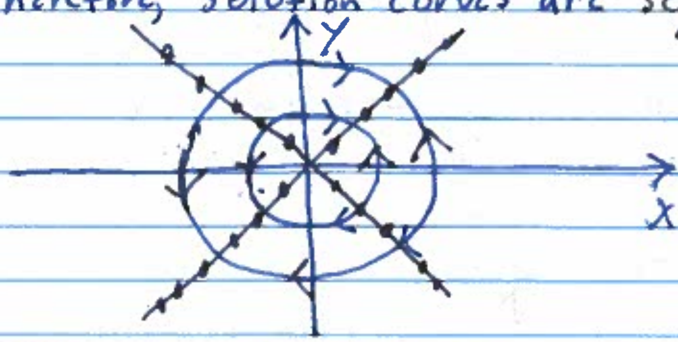
$$\frac{dy}{dx} = \frac{-x}{y}$$

and thus along solution cures

$$\int y\,dy = \int -x\,dx$$

$$\Rightarrow y^2 + x^2 = C.$$

Therefore, solution curves are segments of circles



(b). If we let $x = r\cos\theta$ and $y = r\sin\theta$ then

$$r^2 = x^2 + y^2,$$

$$\tan\theta = y/x.$$

Consequently,

$$r\dot{r} = x\dot{x} + y\dot{y}$$

$$\sec^2\theta\,\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{x^2}$$

$$\Rightarrow \dot{\theta} = \frac{x\dot{y} - y\dot{x}}{x^2}\cos^2\theta = \frac{x\dot{y} - y\dot{x}}{r^2}$$

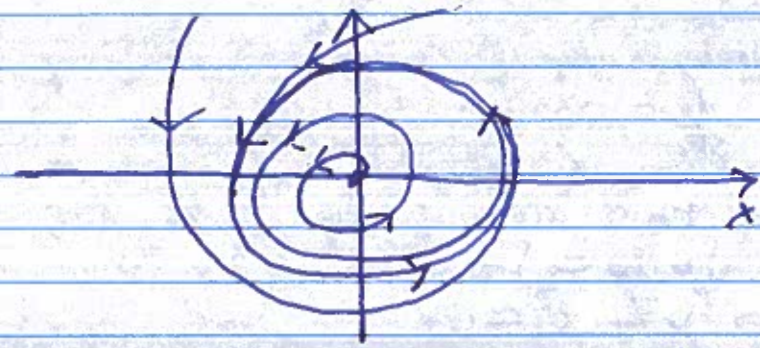
Therefore,

$$\dot{r} = x^2(1-r^2) + y^2(1-r^2)$$

$$\dot{\theta} = x^2 + xy(1-r^2) + y^2 - xy(1-r^2)$$

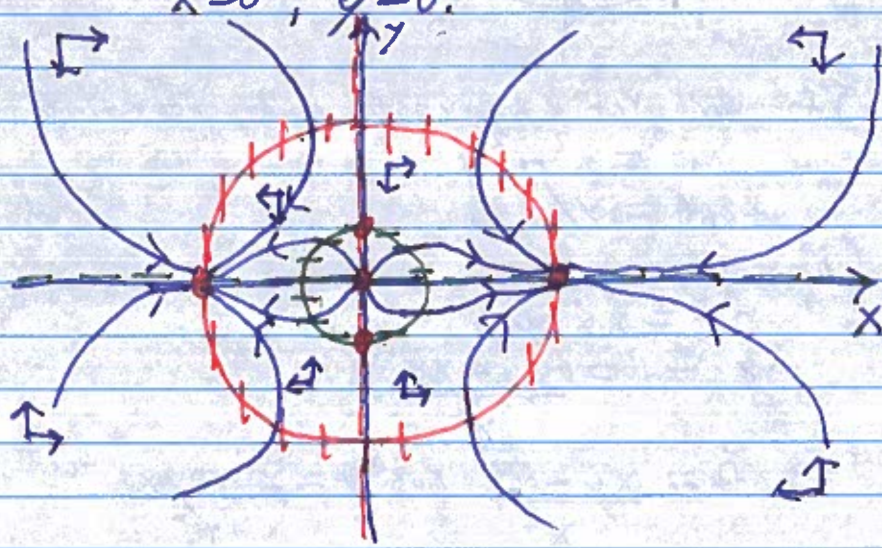
$$\Rightarrow \dot{r} = r^3(1-r^2)$$

$$\dot{\theta} = r^2$$



The closed orbit is given by $r=1$.

(c). The nullclines for this system are the circles
$$x^2+y^2=10,\ x^2+y^2=1$$
as well as the lines
$$x=0,\ y=0.$$



#3.

Consider the following system

$$\dot{x} = x + 3y^2$$
$$\dot{y} = -2x - y$$

(a) Calculate the fixed points and sketch a phase portrait.

(b). Show that the system is conservative.

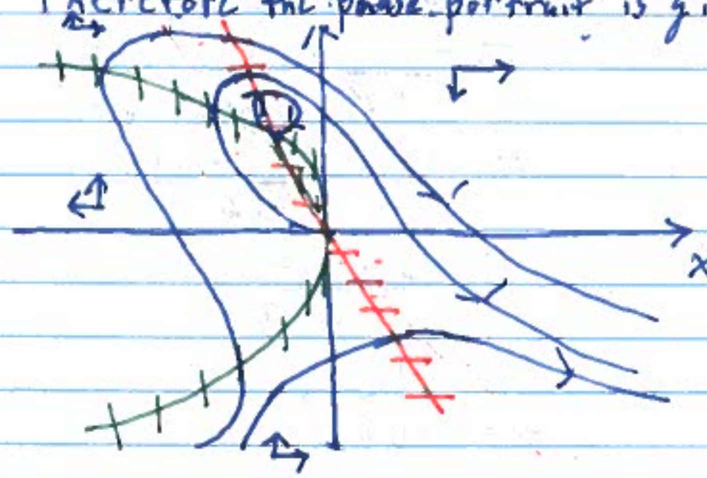
(c). Calculate any homoclinic orbits.

Solution:

(a) The nullclines are given by

$$x = -3y^2.$$
$$y = -2x.$$

Therefore the phase portrait is given by:



(b). If we suppose that this system is conservative then:

$$\frac{dE}{dt} = \frac{\partial E}{\partial x}\dot{x} + \frac{\partial E}{\partial y}\dot{y} = 0$$

If $\frac{\partial E}{\partial x} = \dot{y}$ and $\frac{\partial E}{\partial y} = -\dot{x}$ the system will be conservative. Therefore,

$$\frac{\partial \dot{y}}{\partial y} = -\frac{\partial \dot{x}}{\partial x}.$$

In our case

$$\frac{\partial \dot{y}}{\partial y} = -1 \text{ and } \frac{\partial \dot{x}}{\partial x} = 1$$

and thus this system is conservative. Moreover

$$\frac{\partial E}{\partial x} = -2x-y$$

$$\Rightarrow E = -x^2 - xy + f(y)$$

$$\Rightarrow \frac{\partial E}{\partial y} = -x + f'(y) = -\dot{x} = -x-3y^2$$

Consequently, $f(y) = -y^3$. Therefore,

$$E(x,y) = -x^2 - xy - y^3.$$

The fixed point at which the homoclinic orbit is a part of is located at the point $(-1/12, 1/6)$. Therefore, the homoclinic orbit satisfies:

$$-x^2 - xy - y^3 = -\frac{1}{12^2} + \frac{1}{12\cdot 6} - \frac{1}{6^3}$$

$$= -\frac{1}{6^2\cdot 2^2} + \frac{1}{6\cdot 6\cdot 2} - \frac{1}{6^2\cdot 6}$$

$$= \frac{-6 + 6\cdot 2 - 2\cdot 2}{36\cdot 6}$$

$$= \frac{-3+6-2}{36\cdot 3}$$

$$= \frac{1}{108}.$$

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#4.

Consider the following system in polar coordinates

$$\dot{r} = 0$$

$$\dot{\theta} = (r^2-1)(r^2\cos^2\theta + r\sin\theta + 1)$$

(a). Sketch a phase portrait for this system.

(b). Determine if this system is conservative and find a conserved quantity.

(c) Show that in a neighborhood of the origin the period of any orbit is given by:

$$T(r_0) = 2\pi + ar_0^2 + o(r_0^2).$$

Solution:

Clearly r is constant and the system can be reduced to:

$$\dot{\theta} = (r_0^2-1)(r_0^2\cos^2\theta + r_0\sin\theta + 1).$$

Fixed points satisfy the following equation:

$$r_0^2(1-\sin^2\theta) + r_0\sin\theta + 1$$

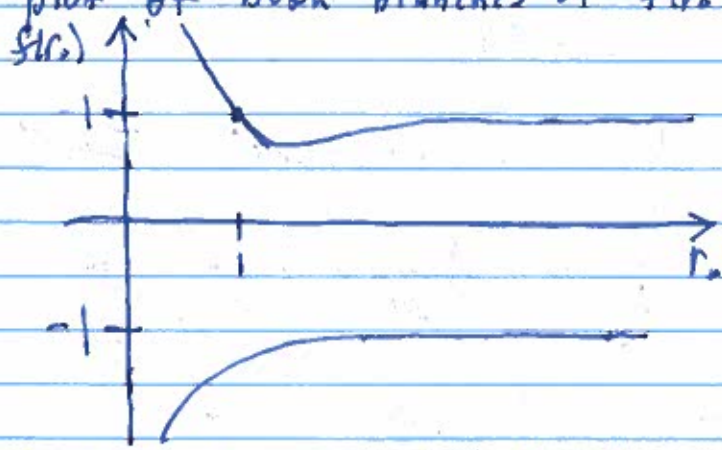
$$\Rightarrow -r_0^2\sin^2\theta + r_0\sin\theta + 1 + r_0^2 = 0$$

$$\Rightarrow \sin\theta = \frac{-r_0 \pm \sqrt{r_0^2 + 4(1+r_0^2)r_0^2}}{2r_0^2}$$

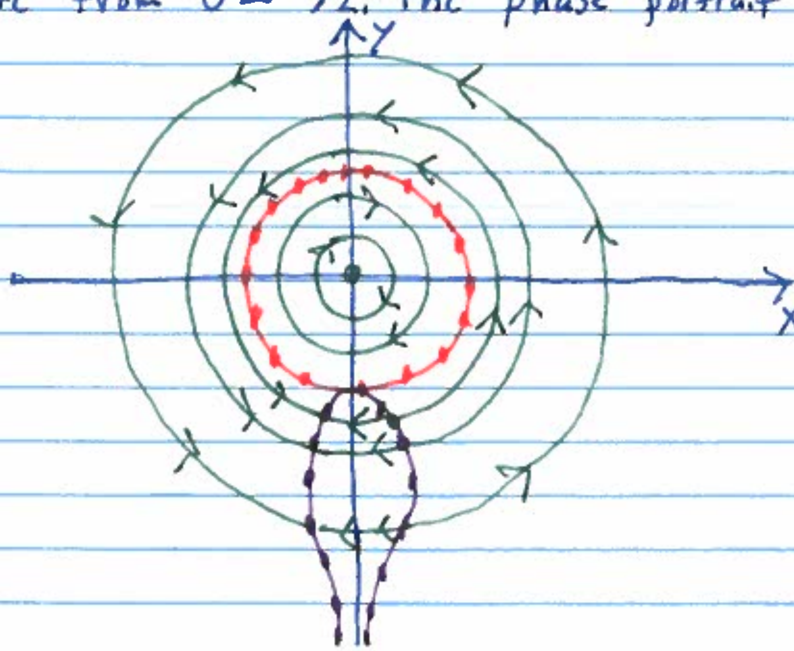
$$= \frac{-1 \pm \sqrt{5+4r_0^2}}{2r_0}$$

$$= f(r_0).$$

The plot of both branches of $f(r_0)$ looks like:



Consequently, there are two fixed points if $r_0 \geq 1$ that bifurcate from $\theta = 3\pi/2$. The phase portrait is therefore



Along a periodic orbit we have that θ varies from 0 to 2π in the clockwise direction and thus

$$\frac{d\theta}{dt} = (r_0^2-1)(r_0^2\cos^2\theta + r_0\sin\theta + 1)$$

$$\Rightarrow \int_0^T dt = \int_0^{2\pi} \frac{1}{(r_0^2-1)(r_0^2\cos^2\theta + r_0\sin\theta+1)}\, d\theta$$

Assuming $r_0 << 1$ we have that

$$-\int_0^{2\pi} \frac{1}{(r_0^2-1)(r_0^2\cos^2\theta + r_0\sin\theta + 1)}\, d\theta$$

$$= \int_0^{2\pi} \frac{1}{(1-r_0^2)(1+r_0\sin\theta + r_0^2\cos^2\theta)}\, d\theta$$

$$= \int_0^{2\pi} (1+r_0^2+r_0^4+\ldots)(1 + (r_0\sin\theta + r_0^2\cos^2\theta) + (r_0\sin\theta + r_0^2\cos^2\theta$$
$$\cdot\, d\theta$$

$$= \int_0^{2\pi} (1 - r_0\sin\theta + r_0^2(1-\cos^2\theta) + r_0^2\sin^2\theta) + r_0^3 f(r_0,\theta))\, d\theta$$

where $f(r_0,\theta)$ is a bounded function for $0<r_0<1$, $0\leq\theta\leq 2\pi$ satisfying $\lim_{r_0\to 0} f(r_0,\theta) \neq 0$. Consequently,

$$T = 2\pi + r_0^2\int_0^{2\pi}(2\sin^2\theta)\, d\theta + o(r_0^2).$$

$$\Rightarrow T = 2\pi(1+r_0^2) + o(r_0^2).$$

$\blacksquare$

#5.

Prove that if A and B are similar matrices then they have the same eigenvalues, and have the same multiplicities.

Solution:

If A and B are similar matrices then there exists nonsingular P such that

$$A = P^{-1}BP.$$

Now, suppose $\vec{v}$ is an eigenvector of A with corresponding eigenvalue λ. Therefore, if we let $\vec{w} = P\vec{v}$ it follows that

$$B\vec{w} = PAP^{-1}\vec{w} = PAP^{-1}P\vec{v} = PA\vec{v} = \lambda P\vec{v} = \lambda\vec{w}.$$

Consequently, every eigenvalue with associated eigenvector of A can be associated with an eigenvalue and eigenvector of B. A similar argument proves the other direction and thus eigenvalues and eigenvectors of A are in 1-1 correspondence between A and B. Moreover,

$$\begin{aligned} \det(\lambda I - A) &= \det(\lambda P^{-1}IP - P^{-1}BP), \\ &= \det(P^{-1}(\lambda I - B)P), \\ &= \det(P^{-1})\det(P)\det(\lambda I - B), \\ &= \det(\lambda I - B). \end{aligned}$$

Consequently, A and B have the same characteristic polynomial and thus algebraic multiplicities. The above bijection proves they have the same geometric multiplicity.