MST 750
Homework #3

Due Date: February 04, 2022

1. For a real $n \times n$ matrix, show that
$$\det(\exp(A)) = \exp(\mathrm{tr}(A)).$$

2. Let $A(t)$, $B(t)$, be differentiable real $n \times n$ matrices.
   (a) Prove that
   $$\frac{d}{dt}A(t)B(t) = \dot{A}(t)B(t) + A(t)\dot{B}(t).$$
   (b) Prove that
   $$\frac{d}{dt}A(t)^{-1} = -A(t)^{-1}\dot{A}(t)A(t)^{-1}.$$

3. Let A, B be $n \times n$ real matrices.
   (a) Find an explicit solution to the equation
   $$\begin{aligned} \dot{x} &= tAx, \\ x(0) &= x_0. \end{aligned}$$
   Hint: Assume A is a scalar and solve this equation and see if the form of the solution generalizes to the matrix case.
   (b) Show that if $[A, [A, B]] = [B, [A, B]] = 0$ then
   $$\exp(At)\exp(Bt) = \exp((A + B)t)\exp([A, B]t^2/2).$$
   Hint: Show that
   $$x(t) = \exp(-(A + B)t)\exp(Bt)\exp(At)x_0$$
   is a solution of the equation
   $$\begin{aligned} \dot{x} &= t[A, B]x \\ x(0) &= x_0. \end{aligned}$$

4. For the following matrices find the explicit solution to the equation
$$\begin{aligned} \dot{x} &= Ax, \\ x(0) &= x_0, \end{aligned}$$
and identify the stable, unstable, and center subspaces.
   (a) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.
   (b) $A = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$.
   (c) $A = \begin{bmatrix} 2 & 1 \\ 0 & -4 \end{bmatrix}$.
   (d) $A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & -2 \\ -1 & 0 & 2 \end{bmatrix}$.

5. Let $f, g : \mathbb{R} \mapsto \mathbb{R}$ be T periodic continuous functions. Show that if the equation

$$\dot{x} = f(t)x$$

has no T-periodic solutions other than the 0 function, then the equation

$$\dot{x} = f(t)x + g(t)$$

has a unique T-periodic solution.

6. pg. 66, #17
7. pg. 66, #18
8. pg. 66, #19