MST 750
Homework #3 #4

Due Date: February 04, 2022

1. For a real $n \times n$ matrix, show that
$$\det(\exp(A)) = \exp(\operatorname{tr}(A)).$$

2. Let $A(t)$, $B(t)$, be differentiable real $n \times n$ matrices.
   (a) Prove that
   $$\frac{d}{dt}A(t)B(t) = \dot{A}(t)B(t) + A(t)\dot{B}(t).$$
   (b) Prove that
   $$\frac{d}{dt}A(t)^{-1} = -A(t)^{-1}\dot{A}(t)A(t)^{-1}.$$

3. Let A, B be $n \times n$ real matrices.
   (a) Find an explicit solution to the equation
   $$\begin{aligned}\dot{x} &= tAx,\\ x(0) &= x_0.\end{aligned}$$
   Hint: Assume A is a scalar and solve this equation and see if the form of the solution generalizes to the matrix case.
   (b) Show that if $[A,[A,B]] = [B,[A,B]] = 0$ then
   $$\exp(At)\exp(Bt) = \exp((A+B)t)\exp([A,B]t^2/2).$$
   Hint: Show that
   $$x(t) = \exp(-(A+B)t)\exp(Bt)\exp(At)x_0$$
   is a solution of the equation
   $$\begin{aligned}\dot{x} &= t[A,B]x\\ x(0) &= x_0.\end{aligned}$$

4. For the following matrices find the explicit solution to the equation
$$\begin{aligned}\dot{x} &= Ax,\\ x(0) &= x_0,\end{aligned}$$
and identify the stable, unstable, and center subspaces.
   (a) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.
   (b) $A = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$.
   (c) $A = \begin{bmatrix} 2 & 1 \\ 0 & -4 \end{bmatrix}$.
   (d) $A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & -2 \\ -1 & 0 & 2 \end{bmatrix}$.

Homework #4

#2.

Let $A(t), B(t)$ be differentiable real $n \times n$ matrices.

(a) Prove that

$$\frac{d}{dt} A(t)B(t) = \dot{A}(t)B(t) + A(t)\dot{B}(t).$$

(b) Prove that

$$\frac{d}{dt} A^{-1} = -A(t)^{-1}\dot{A}(t)A(t)^{-1}.$$

~~proof~~:

$$\left\| \frac{A(t+h)B(t+h) - A(t)B(t)}{h} - \dot{A}(t)B(t) - A(t)\dot{B}(t) \right\|$$

$$= \left\| \left(\frac{A(t+h) - A(t)}{h} - \dot{A}(t) \right) B(t) - \frac{A(t+h)B(t)}{h} + \frac{A(t+h)B(t+h)}{h} - A(t)\dot{B}(t) \right\|$$

$$= \left\| \left(\frac{A(t+h) - A(t)}{h} - \dot{A}(t) \right) B(t) + A(t+h) \left(\frac{B(t+h) - B(t)}{h} - \dot{B}(t) \right) + (A(t+h) - A(t))\dot{B}(t) \right\|$$

$$\leq \left\| \left(\frac{A(t+h) - A(t)}{h} - \dot{A}(t) \right) \right\| \cdot \|B(t)\| + \|A(t+h)\| \cdot \left\| \frac{B(t+h) - B(t)}{h} - \dot{B}(t) \right\| + \|A(t+h) - A(t)\| \cdot \|\dot{B}(t)\|$$

Since A and B are differentiable they are continuous and thus

$$\lim_{h \to 0} \left\| \frac{A(t+h) - A(t)}{h} - \dot{A}(t) \right\| = \lim_{h \to 0} \left\| \frac{B(t+h) - B(t)}{h} - \dot{B}(t) \right\| = 0,$$

$$\lim_{h \to 0} \|A(t+h)\| = \|A(t)\|, \quad \lim_{h \to 0} \|A(t+h) - A(t)\| = 0.$$

Therefore,

$$\lim_{h \to 0} \left\| \frac{A(t+h)B(t+h) - A(t)B(t)}{h} - \dot{A}(t)B(t) - A(t)\dot{B}(t) \right\| = 0$$

and thus

$$\frac{d}{dt} A(t)B(t) = \dot{A}(t)B(t) + A(t)\dot{B}(t).$$

Consequently, since

$$A^{-1}(t)A(t) = I$$

it follows that

$$\frac{d}{dt}A^{-1}(t)A(t) = 0$$

$$\Rightarrow \dot{A}^{-1}(t)A(t) + A^{-1}\dot{A}(t) = 0$$

$$\Rightarrow \quad \dot{A}^{-1}(t) = -A^{-1}(t)\dot{A}(t)A^{-1}(t).$$

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#4.

For the following matrices find the explicit solution to the equation

$$\dot{x} = Ax,$$

$$x(0) = x_0,$$

and identify the stable, unstable, and center subspaces.

(c) $A = \begin{bmatrix} 2 & 1 \\ 0 & -4 \end{bmatrix}$,

(d) $A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & -2 \\ -1 & 0 & 2 \end{bmatrix}$.

(c) The eigenvalues are clearly $\lambda_1 = 2$, $\lambda_2 = -4$. One eigenvector is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ corresponding to $\lambda_1 = 2$. To find the other eigenvector note that

$$-4I - A = \begin{bmatrix} -6 & -5 \\ 0 & 0 \end{bmatrix}$$

Thus the other eigenvector is given by:

$$\begin{bmatrix} -5/6 \\ 1 \end{bmatrix}$$

Consequently, the explicit solution is given by:

$$\vec{x}(t) = \begin{bmatrix} 2 & -5/6 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} e^{+2t} & 0 \\ 0 & e^{-4t} \end{bmatrix} \begin{bmatrix} 2 & -5/6 \\ 0 & -4 \end{bmatrix}^{-1} \vec{x}_0.$$

$E_u = \operatorname{span}\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\}$, $E_s = \operatorname{span}\{\begin{bmatrix} -5/6 \\ 1 \end{bmatrix}\}$.

(d) For this matrix

$$\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda-2 & 0 & -1 \\ -1 & \lambda-2 & 2 \\ 1 & 0 & \lambda-2 \end{bmatrix}\right)$$

$$= (\lambda-2)\det\left(\begin{bmatrix} \lambda-2 & -1 \\ 1 & \lambda-2 \end{bmatrix}\right)$$

$$= (\lambda-2)(\lambda^2-4\lambda+5)$$

Therefore, the eigenvalues are given by

$$\lambda_1 = 2$$

$$\lambda_2 = \frac{4+\sqrt{16-20}}{2} = 2+i$$

$$\lambda_3 = \lambda_2^*.$$

Consequently, $E_v = \mathbb{R}^3$. To calculate eigenvectors, we have that

$$2I - A = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 2 \\ 1 & 0 & 0 \end{bmatrix}$$

The corresponding eigenvector is therefore $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. We also have that

$$(2+i)I - A = \begin{bmatrix} i & 0 & -1 \\ -1 & i & 2 \\ 1 & 0 & i \end{bmatrix}$$

$$\Rightarrow ix - z = 0$$

$$-x + iy + 2z = 0$$

$$\Rightarrow z = ix$$

$$\Rightarrow iy + (2i-1)x = 0$$

$$\Rightarrow y = (-2-i)x$$

Therefore, the corresponding eigenvector is

$$\vec{v} = \begin{bmatrix} 1 \\ -2-i \\ i \end{bmatrix}$$

$$\Rightarrow \operatorname{Re}(\vec{v}) = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \operatorname{Im}(\vec{v}) = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

It follows that

$$\vec{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 & 0 \\ 0 & e^{2t}\cos(t) & -e^{2t}\sin(t) \\ 0 & e^{2t}\sin(t) & e^{2t}\cos(t) \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & -1 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \vec{x}_0.$$

$\blacksquare$

#5.

Let $f, g : \mathbb{R} \to \mathbb{R}$ be T periodic continuous functions. Show that if the equation

$$\dot{x} = f(t)x$$

has no periodic solutions other than the 0-function, then the equation

$$\dot{x} = f(t)x + g(t)$$

has a unique T-periodic solution.

Solution:

We first show that this equation has a periodic solution. Assuming $x(t_0) = x_0$ we have that

$$\dot{x} - f(t)x = g(t)$$

$$\Rightarrow e^{-\int_{t_0}^{t} f(s)ds}(\dot{x} - f(t)x) = e^{-\int_{t_0}^{t} f(s)ds} g(t)$$

$$\Rightarrow \frac{d}{dt}\left(e^{-\int_{t_0}^{t} f(s)ds} x\right) = e^{-\int_{t_0}^{t} f(s)ds} g(t)$$

$$\Rightarrow \int_{x_0}^{e^{-\int_{t_0}^{t} f(s)ds}} d\left(e^{-\int_{t_0}^{t} f(s)ds} x\right) = \int_{t_0}^{t} e^{-\int_{t_0}^{t} f(s)ds} g(t)dt.$$

$$\Rightarrow x(t) = x_0 A(t, t_0) + \int_{t_0}^{t} A(t, s)g(s)ds,$$

where

$$A(t, s) = \exp\left(\int_s^t f(s)ds\right).$$

If we consider the equation

$$x(t_0 + T) = x(t_0)$$

we have that

$$x_0 = x_0 A(t_0 + T, t_0) + \int_{t_0}^{t_0+T} A(t_0 + T) g(s)ds.$$

Define $F: \mathbb{R} \to \mathbb{R}$ by

$$F(x_0) = x_0(1 - A(t_0+T, t_0)) - \int_{t_0}^{t_0+T} A(t_0+T)g(s)ds.$$

Therefore,

$$-\lim_{x_0 \to \infty} F(x_0) = \operatorname{sgn}(1 - A(t_0+T), t_0)\infty,$$

$$-\lim_{x_0 \to -\infty} F(x_0) = -\operatorname{sgn}(1 - A(t_0+T), t_0)\infty.$$

and thus by the intermediate value theorem there exists x_0 such that $F(x_0) = 0$. Therefore, there exists a periodic solution.

Now, suppose x_1^*, x_2^* are periodic solution and let $x^* = x_2^* - x_1^*$. It follows that x^* is periodic and satisfies the O.D.E

$$\begin{aligned} \dot{x}^* &= \dot{x}_2^* - \dot{x}_1^* \\ &= f(t)x_2^* + g(t) - f(t)x_1^* - g(t) \\ &= f(t)x^*. \end{aligned}$$

Consequently, since $x = 0$ is the only periodic solution of the above equation it follows that $x^* = 0$ and thus $x_1^* = x_2^*$.

$\blacksquare$

#6

Discuss solutions to the equation

$$\dot{x} = Ax + f(t)$$
$$x(0) = x_0.$$

Solution:

If x is a solution it follows that

$$e^{-At}\dot{x} - e^{-At}x = e^{-At}f(t)$$

$$\Rightarrow \frac{d}{dt}\left(e^{-At}x\right) = e^{-At}f(t)$$

$$\Rightarrow \int_{x_0}^{e^{-At}x} d(e^{-At}x) = \int_0^t e^{-As} f(s)ds$$

$$\Rightarrow x = e^{At}x_0 + \int_0^t e^{A(t-s)}f(s)ds.$$

Now, if $f=b$ a constant, it follows that

$$x(t) = e^{At}x_0 + e^{At}\int_0^t e^{-As}b\,ds$$
$$= e^{At}x_0 + e^{At}(A^{-1}e^{-At} - A^{-1})b$$
$$= e^{At}(x_0 - A^{-1}e^{-At}b) + A^{-1}b$$

This, of course, assumes A is invertible and is equivalent to changing to a coordinate system $y = x + A^{-1}b$. Moreover, if $b \in Range(A)$ then the above formal calculation still works.

If, however, $b \notin Range(A)$ then A^{-1} cannot be defined. The difficulty arises in computing $\int_0^t e^{-As}ds$. If A is singular it has a block diagonal representation:

$$A = P\begin{bmatrix} \begin{matrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 \ddots & \\ & & & 0\,1 \\ & & & 0 \end{matrix} & 0 \\ 0 & A_B \end{bmatrix}P^{-1}$$

where A_B is full rank. Therefore,

$$e^{At} = P\begin{bmatrix} \begin{matrix} 1 & t & t^2 & \cdots \\ & \ddots & t & \cdots \\ & & \ddots & \\ & & & 1 \end{matrix} & 0 \\ 0 & A_B \end{bmatrix}$$

$$\Rightarrow \int_0^t e^{At} = P\begin{bmatrix} \begin{matrix} t & t^2/2 & t^3/3 & \cdots \\ 0 & \ddots & & \\ & & \ddots & \\ & & & t \end{matrix} & \\ & e^{A_B} \end{bmatrix}P^{-1}.$$

#7

Show that the naive solution to

$$\frac{d}{dt}\Phi = A(t)\Phi, \quad \Phi(0,0) = I$$

does not work unless $[A(s), A(t)] = 0$ for all $s,t \in \mathbb{R}$.

Solution:

The naive solution is given by:

$$\Phi(t,0) = \exp\left(\int_0^t A(s)\,ds\right)$$

Expanding, we have to quadratic order that:

$$\Phi(t,0) = I + \int_0^t A(s)\,ds + \tfrac{1}{2}\left(\int_0^t A(s)\,ds\right)\left(\int_0^t A(u)\,du\right) + \dots$$

$$\Rightarrow \Phi'(t,0) = A(t) + \tfrac{1}{2}\left(A(t)\int_0^t A(s)\,ds + \left(\int_0^t A(s)\,ds\right)A(t)\right) + \dots$$
$$= A(t) + \tfrac{1}{2}\int_0^t \left(A(t)A(s) + A(s)A(t)\right)ds + \dots$$

However,

$$A(t)\Phi = A(t) + \int_0^t A(t)A(s)\,ds + \dots$$

These expansions will not much unless $[A(t), A(s)] = 0$.
However if $[A(t), A(s)] = 0$ then.

$$\Phi'(t,0) = A(t) + \int_0^t A(t)A(s)\,ds + \tfrac{1}{2}\int_0^t\int_0^t A(t)A(s)A(u)\,ds\,du$$
$$+ \dots$$
$$= A(t)\left(I + \int_0^t A(s)\,ds + \tfrac{1}{2}\int_0^t\int_0^t A(s)A(u)\,ds\,du + \dots\right)$$
$$= A(t)\,\Phi(t,0).$$

$\blacksquare$

#8

Discuss solutions to the ODE:

$$\dot{x} = A(t)x,$$

with

$$A(t) = \begin{pmatrix} 1 & t \\ 0 & -1 \end{pmatrix}.$$

Solution:

Computing

$$[A(t), A(s)] = \begin{bmatrix} 1 & t \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & s \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & s \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & t \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & s-t \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & t-s \\ 0 & 1 \end{bmatrix} \neq 0.$$

Now, if we let

$$B(t) = \int_0^t A(s)\,ds$$

it follows that

$$B(t) = \begin{bmatrix} t & t^2/2 \\ 0 & -t \end{bmatrix} = t\begin{bmatrix} 1 & t/2 \\ 0 & -1 \end{bmatrix}.$$

Consequently,

$$B(t)^2 = t^2\begin{bmatrix} 1 & t/2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & t/2 \\ 0 & -1 \end{bmatrix} = t^2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$B(t)^3 = t^3\begin{bmatrix} 1 & t/2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = t^3\begin{bmatrix} 1 & t/2 \\ 0 & -1 \end{bmatrix}$$

$$\vdots$$

Therefore,

$$\exp(B(t)) = \begin{bmatrix} 1 + t + t^2/2 + \dots & 0 + t/2\,(t + 1/3!\,t^3 + \dots) \\ 0 & 1 - t + t^2/2 - t^3/6 + \dots \end{bmatrix},$$

$$= \begin{bmatrix} e^t & t/2\,\sinh(t) \\ 0 & e^{-t} \end{bmatrix},$$

which does not satisfy the O.D.E.

We can solve these equations explicitely

$$\dot{x} = x + ty$$

$$\dot{y} = -y$$

$$\Rightarrow y = e^{-t} y_0$$

$$\Rightarrow \dot{x} = x + te^{-t} y_0$$

$$\Rightarrow \dot{x} - x = te^{-t} y_0$$

$$\Rightarrow e^{-t}\dot{x} - e^{-t}x = ty_0$$

$$\Rightarrow \frac{d}{dt} xe^{-t} = ty_0$$

$$\Rightarrow \int_{x_0}^{xe^{-t}} d(xe^{-t}) = \int_0^t ty_0$$

$$\Rightarrow xe^{-t} = x_0 + t^2/2\, y_0$$

$$\Rightarrow x(t) = x_0 e^t + \frac{t^2}{2} e^t y_0$$

∎