MST 750
Homework #3

Due Date: February 28, 2022

1. pg. 97, #1.

2. pg. 97, #3.

3. Let I be a closed and bounded interval in $\mathbb{R}$ containing 0.

   (a) Let $Y \subset C^0(I;\mathbb{R})$ be the space of Lipschitz continuous functions with Lipschitz constant $L > 0$. Show that Y is a Banach space with respect to the norm $\|f\|_\infty = \sup_{t\in I}\{|f(t)|\}$.

   (b) Let $Z \subset Y$ be defined by $Z = \{f \in Y : f(0) = 0\}$. Show that the mapping $\|\cdot\|_Z : Z \mapsto \mathbb{R}$ defined by
   $$\|f\|_Z = \sup_{t\in I\setminus\{0\}} \left\{\frac{|f(t)|}{|t|}\right\}$$
   defines a norm on Z.

   (c) Prove for all $f, g \in Z$ that $\|f - g\|_Z \leq L$.

   (d) Prove that Z with respect to the norm $\|\cdot\|_Z$ is a Banach space.

4. In this problem you will need the following definition. A map $T : X \mapsto Y$ between two Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ is continuous if for all $x_n \in X$ satisfying $x_n \to x^*$, i.e. $\lim_{n\to\infty}\|x_n - x^*\|_X = 0$, it follows that $T(x_n) \to T(x)$, i.e. $\lim_{n\to\infty}\|T(x_n) - T(x)\|_Y = 0$.

   (a) If X is Banach space prove for all $f, g \in X$ that $|\|f\| - \|g\|| < \|f - g\|$.

   (b) If X is Banach space prove that the norm on X is continuous when viewed as a map from X to $\mathbb{R}$. That is, prove that the map $T : X \mapsto \mathbb{R}$ defined by $T(f) = \|f\|$ is continuous.

   (c) If X, Y are Banach spaces with the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ prove that the product space $X \times Y$ with the norm $(f, g) = \|f\|_X + \|g\|_Y$ is a Banach space.

   (d) If X is a Banach space prove that the transformation $T : X \times X \mapsto X$ defined by $T(f, g) = f + g$ is a continuous map from $X \times X$ into X.

   (e) If X a Banach space prove that the map $T : X \times \mathbb{R} \mapsto X$ defined by $T(f, a) = af$ is continuous.

5. Let $T : X \mapsto X$ be a transformation of a Banach space X such that T^m is a contraction for some $m \in \mathbb{N}$. Show that:

   (a) T has a unique fixed point $x_0 \in X$, i.e. there exists a unique $x_0 \in X$ such that $T(x_0) = x_0$.

   (b) For each $x \in X$ the sequence $T^n(x)$ converges to x_0 when $n \to \infty$.

6. pg. 98, #4.

7. pg. 99, #7. There is a typo in the book. The first guess should be $u_0(t) = a$.

8. Consider the following first order system.
   $$\dot{x} = 2t - 2\sqrt{\max\{0, x\}},$$
   $$x(0) = 0.$$
   Apply Picard iteration with the initial guess $x_0 = 0$. Explicitly find the pattern for the iterations. Do the iterations converge?