MST 750

Homework ~~#3~~ #5

Due Date: February 28, 2022

1. pg. 97, #1.
2. pg. 97, #3.
3. Let I be a closed and bounded interval in $\mathbb{R}$ containing 0.
   (a) Let $Y \subset C^0(I;\mathbb{R})$ be the space of Lipschitz continuous functions with Lipschitz constant $L > 0$. Show that Y is a Banach space with respect to the norm $\|f\|_\infty = \sup_{t\in I}\{|f(t)|\}$.
   (b) Let $Z \subset Y$ be defined by $Z = \{f \in Y : f(0) = 0\}$. Show that the mapping $\|\cdot\|_Z : Z \mapsto \mathbb{R}$ defined by
   $$\|f\|_Z = \sup_{t\in I\setminus\{0\}} \left\{\frac{|f(t)|}{|t|}\right\}$$
   defines a norm on Z.
   (c) Prove for all $f, g \in Z$ that $\|f - g\|_Z \leq L$.
   (d) Prove that Z with respect to the norm $\|\cdot\|_Z$ is a Banach space.
4. In this problem you will need the following definition. A map $T : X \mapsto Y$ between two Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ is continuous if for all $x_n \in X$ satisfying $x_n \to x^*$, i.e. $\lim_{n\to\infty}\|x_n - x^*\|_X = 0$, it follows that $T(x_n) \to T(x)$, i.e. $\lim_{n\to\infty}\|T(x_n) - T(x)\|_Y = 0$.
   (a) If X is Banach space prove for all $f, g \in X$ that $|\|f\| - \|g\|| < \|f - g\|$.
   (b) If X is Banach space prove that the norm on X is continuous when viewed as a map from X to $\mathbb{R}$. That is, prove that the map $T : X \mapsto \mathbb{R}$ defined by $T(f) = \|f\|$ is continuous.
   (c) If X, Y are Banach spaces with the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ prove that the product space $X \times Y$ with the norm $(f, g) = \|f\|_X + \|g\|_Y$ is a Banach space.
   (d) If X is a Banach space prove that the transformation $T : X \times X \mapsto X$ defined by $T(f, g) = f + g$ is a continuous map from $X \times X$ into X.
   (e) If X a Banach space prove that the map $T : X \times \mathbb{R} \mapsto X$ defined by $T(f, a) = af$ is continuous.
5. Let $T : X \mapsto X$ be a transformation of a Banach space X such that T^m is a contraction for some $m \in \mathbb{N}$. Show that:
   (a) T has a unique fixed point $x_0 \in X$, i.e. there exists a unique $x_0 \in X$ such that $T(x_0) = x_0$.
   (b) For each $x \in X$ the sequence $T^n(x)$ converges to x_0 when $n \to \infty$.
6. pg. 98, #4.
7. pg. 99, #7. There is a typo in the book. The first guess should be $u_0(t) = a$.
8. Consider the following first order system.
   $$\dot{x} = 2t - 2\sqrt{\max\{0, x\}},$$
   $$x(0) = 0.$$
   Apply Picard iteration with the initial guess $x_0 = 0$. Explicitly find the pattern for the iterations. Do the iterations converge?

Homework #5

#3

Let $\mathbb{I}$ be a closed and bounded interval containing 0.

(a) Let $\mathbb{Y} \subset C^0(\mathbb{I};\mathbb{R})$ be Lipschitz continuous functions with Lipschitz constant $L > 0$. Show that $\mathbb{Y}$ is a Banach space with respect to the norm
$$\|f\|_\infty = \sup_{t\in\mathbb{I}} \{|f(t)|\}.$$

(b) Let $Z \subset \mathbb{Y}$ be defined by $Z = \{f \in \mathbb{Y} : f(0) = 0\}$. Show the the mapping $\|\cdot\|_Z : Z \to \mathbb{R}$ defined by
$$\|f\|_Z = \sup_{t\in\mathbb{I}\setminus\{0\}} \left\{\frac{|f(t)|}{|t|}\right\}$$
defines a norm on Z.

(c) Prove for all $f, g \in Z$ that
$$\|f-g\|_Z \leq L.$$

(d) Prove that Z with respect to the norm $\|\cdot\|_Z$ is a Banach space.

~~proof~~:

(a). Given that $C^0(\mathbb{I};\mathbb{R})$ is a Banach space with respect to $\|\cdot\|_\infty$ we just need to show that if $f_n \in \mathbb{Y}$ is Cauchy that it converges to a point in $\mathbb{Y}$. If $f_n \in \mathbb{Y}$ is Cauchy then there exists $f^* \in C^0(\mathbb{I};\mathbb{R})$ such that $f_n \to f^*$. Furthermore,
$$|f^*(x) - f^*(y)| = \lim_{n\to\infty} |f_n(x) - f_n(y)|$$
$$\leq L|x-y|$$
and thus $f^* \in \mathbb{Y}$.

(b.) I am just going to look at the triangle inequality since the other properties are obvious. For $f, g, \in Z$ follows that
$$\|f-g\|_Z = \sup_{t\in\mathbb{I}\setminus\{0\}} \left\{\frac{|f(t)-g(t)|}{t}\right\}$$

$$\Rightarrow \|f-g\|_Z \le \sup_{t\in I\setminus\{0\}} \left\{ \frac{|f(t)|+|g(t)|}{|t|} \right\}$$
$$\le \sup_{t\in I\setminus\{0\}} \left\{ \frac{|f(t)|}{|t|} \right\} + \sup_{t\in I\setminus\{0\}} \frac{|g(t)|}{|t|}$$
$$= \|f\|_Z + \|g\|_Z$$

(c). If, $f, g \in Z$ then
$$\|f-g\|_Z = \sup_{t\in I\setminus\{0\}} \left\{ \frac{|f(t)-g(t)|}{|t|} \right\}$$
$$\le \sup_{t\in I\setminus\{0\}} \left\{ \frac{|f(t)-f(0)|}{|t|} + \frac{|g(t)-g(0)|}{|t|} \right\}$$
$$\le \sup_{t\in I\setminus\{0\}} \left\{ \frac{L|t|}{|t|} + \frac{L|t|}{|t|} \right\}$$
$$= 2L.$$

(d). Let $f_n \in Z$ be Cauchy. Therefore, for all $t \neq 0$ it follows that for all $m,n \in \mathbb{N}$
$$\frac{|f_m(t)-f_n(t)|}{|t|} \le \|f_m - f_n\|_Z$$
$$\Rightarrow |f_m(t)-f_n(t)| \le \|f_m-f_n\|_Z |t|.$$
Consequently, $f_m(t)$ is Cauchy as a sequence of real numbers. Therefore, there exist $f^*: I \to \mathbb{R}$ defined by
$$f^*(t) = \begin{cases} 0, & t=0 \\ \lim_{n\to\infty} f_n(t), & t\neq 0. \end{cases}$$

Moreover, since f_n is Cauchy, for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $m,n \ge N$ implies
$$\|f_m - f_n\|_Z < \varepsilon$$
$$\Rightarrow \frac{|f_m(t)-f_n(t)|}{t} < \varepsilon \quad (t \neq 0).$$
Taking the limit as $m \to \infty$ we obtain
$$\frac{|f^*(t)-f_n(t)|}{t} < \varepsilon$$
$$\Rightarrow \sup_{t\in I\setminus\{0\}} \frac{|f^*(t)-f_n(t)|}{t} < \varepsilon.$$

We finally need to prove that f^* is Lipschitz Continuous. By continuity of $|\cdot|$ it follows that

$$|f^*(t_1)-f^*(t_2)| = \lim_{n\to\infty} |f_n(t_1)-f_n(t_2)| \leq L\cdot|t_1-t_2|.$$

■

#4.

(a) If X is a Banach space prove for all $f,g\in X$ that
$$\big|\|f\|-\|g\|\big| \leq \|f-g\|.$$

(b) Prove that $T: X\to\mathbb{R}$ defined by $T(f)=\|f\|$ is continuous.

(c) Prove that the product space $X\times Y$ with the natural norm is a Banach space.

(d) Prove that the map $T: X\times X\to X$ defined by $T(f,g)=f+g$ is continuous.

(e) Prove that the map $T: X\times\mathbb{R}\to X$ defined by $T(f,a)=af$ is continuous.

~~proof~~:

(a) For $f,g\in X$ it follows that
$$\|f\| = \|f-g+g\| \leq \|f-g\|+\|g\|$$
$$\Rightarrow \|f\|-\|g\| \leq \|f-g\|.$$
Also,
$$\|g\| = \|g-f+f\| \leq \|f-g\|+\|f\|$$
$$\Rightarrow \|g\|-\|f\| \leq \|f-g\|.$$
Therefore,
$$\big|\|f\|-\|g\|\big| = \max\{\|f\|-\|g\|,\ \|g\|-\|f\|\} \leq \|f-g\|.$$

(b). Suppose $f_n\to f$. Therefore,
$$|T(f)-T(f_n)| = \big|\|f\|-\|f_n\|\big| \leq \|f_n-f\|$$
and thus $T(f_n)\to T(f)$.

(c). Suppose $(f_n,g_n)\in X,Y$ is Cauchy with respect to the norm $\|(f,g)\| = \|f\|_X+\|g\|_Y$. Therefore, f_n,g_n are Cauchy in X,Y and thus there exists f^*,g^* such that $f_n\xrightarrow{X} f^*$, $g_n\xrightarrow{Y} g^*$. Moreover,
$$\lim_{n\to\infty}\|(f_n-f^*,g_n-g^*)\| = \lim_{n\to\infty}\left(\|f_n-f^*\|_X+\|g_n-g^*\|_Y\right)=0.$$

and thus $\mathbb{X}\times\mathbb{X}$ is complete.

(d) Suppose $(f_n, g_n)\to(f^*, g^*)$ in $\mathbb{X}\times\mathbb{X}$. Therefore,

$$\|T(f_n,g_n)-T(f,g)\| = \|f_n+g_n-f-g\|,$$
$$\leq \|f_n-f\|+\|g_n-g\|,$$
$$= \|(f_n g_n)-(f,g)\|.$$

Therefore,

$$\lim_{n\to\infty} T(f_n,g_n) = T(f,g)$$

(e). Suppose $(a_n, f_n)\to(a^*, f^*)$ in $\mathbb{R}\times\mathbb{X}$. Therefore,

$$\|T(a_n,f_n)-T(a^*,f^*)\| = \|a_n f_n - a^* f^*\|$$
$$= \|a_n f_n - a_n f^*\| + \|a_n f^* - a^* f^*\|$$

Now, since $(a_n, f_n)\to(a^*, f^*)$ it follows that

$$\lim_{n\to\infty} |a_n-a^*| + \|f_n-f^*\| = 0$$

and thus a_n is bounded. Therefore, there exists $M>0$ such that

$$\|T(a_n,f_n)-T(a^*,f^*)\| \leq M\|f_n-f^*\| + |a_n-a^*|\,\|f^*\|$$

$$\leq \max\{M, \|f^*\|\}\cdot\|(a_n-a^*, f_n-f^*)\|$$

Thus,

$$\lim_{n\to\infty} \|T(a_n,f_n)-T(a^*,f^*)\| = 0.$$

■

#5.

Let $T:\mathbb{X}\to\mathbb{X}$ be a transformation of a Banach space $\mathbb{X}$ such that T^m is a contraction for some $m\in\mathbb{N}$. Show that

(a) T has a unique fixed point.

(b) For each $x\in\mathbb{X}$ the sequence $T^n(x)$ converges to x_0 when $n\to\infty$.

Solution:

Define a sequence x_n by

$$x_n = T^n(x_0) = T(x_{n-1}).$$

and the map $S:\mathbb{X}\to\mathbb{X}$ by $S(x) = T^m(x)$.

Therefore, for all $n \in \mathbb{N}$ it follows that

$$T^n(x) = S^{q_n} \circ T^{r_n}(x) = T^{r_n} \circ S^{q_n}(x)$$

where $q_n = \lfloor n/m \rfloor$ and $0 \le r_n < m$. Consequently, for $a, b \in \mathbb{N}$ satisfying $a \ge b$ it follows that

$$\begin{aligned}\|T^a(x_0) - T^b(x_0)\| &= \|S^{q_a}(T^{r_a}(x_0)) - S^{q_b}(T^{r_b}(x_0))\| \\ &\le c^{q_b}\|S^{q_a - q_b}(T^{r_a}(x_0)) - T^{r_b}(x_0)\| \\ &= c^{q_b}\|S^{q_a-q_b}(T^{r_a}(x_0)) - S^{q_a-q_b-1}(T^{r_a}(x_0)) + S^{q_a-q_b-1}(T^{r_a}(x_0)) \\ &\quad + \dots + S(T^{r_a}(x_0)) - T^{r_b}(x_0)\| \\ &\le c^{q_b}(c^{q_a-q_b-1} + \dots + c + 1)M \\ &\le \frac{c^{q_b}}{1-c}M\end{aligned}$$

where $M = \max_{0 \le r \le m} \{\|S(T^r(x_0)) - T^r(x_0)\|\}$. Therefore, x_n is Cauchy and thus there exists x^* such that $\lim_{n\to\infty} x_n = x^*$.

Now, by continuity of S it follows that

$$\begin{aligned}x^* &= \lim_{n\to\infty} x_n \\ &= \lim_{n\to\infty} S(x_{n-m}) \\ &= S(\lim_{n\to\infty} x_{n-m}) \\ &= S(x^*).\end{aligned}$$

That is, x^* is a fixed point of S and is unique since S is a contraction. Consequently,

$$x^* = \lim_{n\to\infty} T^{nm+1}(x^*) = \lim_{n\to\infty} T(S^n(x^*)) = \lim_{n\to\infty} T(x^*) = T(x^*).$$

To show uniqueness suppose both x_0, y_0 satisfy $T(x_0) = x_0$ and $T(y_0) = y_0$. Therefore,

$$\|x_0 - y_0\| = \|T(x_0) - T(y_0)\| = \|T^m(x_0) - T^m(y_0)\| < c\|x_0 - y_0\|$$

and consequently $x_0 = y_0$. ∎

#6.

Consider the operator $T: C^0([-1,1];\mathbb{R}) \to C^0([-1,1];\mathbb{R})$ defined by

$$T(f) = \sin(2\pi x) + \lambda \int_{-1}^{1} \frac{f(y)}{1+(x-y)^2}\,dy$$

equipped with the sup-norm

(a) Show that if $f \in C^0([-1,1];\mathbb{R})$ then so is $T(f)$.

(b) Find λ_0 such that if $|\lambda| < \lambda_0$, then $T(f)$ is a contraction and if $|\lambda| > \lambda_0$, then it is not.

Solution:

(a) For $x_1, x_2 \in [-1,1]$ it follows that

$$|Tf(x_1) - Tf(x_2)| = \left| \sin(2\pi x_1) - \sin(2\pi x_2) + \lambda \int_{-1}^{1} \left(\frac{f(y)}{1+(x_1-y)^2} - \frac{f(y)}{1+(x_2-y)^2} \right) dy \right|$$

$$\leq 2\pi |x_2 - x_1| + \lambda \int_{-1}^{1} \frac{|f(y)|\,|x_2^2 - 2y(x_2-x_1) - x_1^2|}{(1+(x_1-y)^2)(1+(x_2-y)^2)}\,dy$$

$$\leq 2\pi |x_2 - x_1| + \lambda \|f\|_\infty \int_{-1}^{1} \frac{(|x_2-x_1||x_2+x_1| + 2|y||x_2-x_1|)}{(1+(x_1-y)^2)(1+(x_2-y)^2)}\,dy$$

$$\leq 2\pi |x_2 - x_1| + \lambda \|f\|_\infty \int_{-1}^{1} (2|x_2-x_1| + 2|x_2-x_1|)\,dy$$

$$\leq 2\pi |x_2 - x_1| + \lambda \|f\|_\infty 8 |x_2 - x_1|$$

and thus Tf is Lipschitz continuous.

(b). For $f, g \in C^0([-1,1];\mathbb{R})$ it follows that

$$|Tf(x) - Tg(x)| = |\lambda| \left| \int_{-1}^{1} \frac{f(y)-g(y)}{1+(x-y)^2}\,dy \right|$$

$$\leq |\lambda| \int_{-1}^{1} \frac{|f(y)-g(y)|}{1+(x-y)^2}\,dy$$

$$\leq |\lambda| \, \|f-g\|_\infty (\tan^{-1}(1+x) + \tan^{-1}(1-x))$$

$$\leq |\lambda| \, \|f-g\|_\infty (\tan^{-1}(1) + \tan^{-1}(1))$$

$$= |\lambda| \, \|f-g\|_\infty \, \pi/2.$$

Therefore, if $|\lambda| < 2/\pi$ this map is a contraction.

Furthermore, if $f=1$ and $g=0$ it follows that

$$|Tf(x)-Tg(x)| = |\lambda| \left| \int_{-1}^{1} \frac{1}{1+(x-y)^2}\, dy \right|$$

$$= |\lambda| \left| \tan^{-1}(1+x) + \tan^{-1}(1-x) \right|$$

$$\Rightarrow \|Tf - Tg\|_\infty = |\lambda| \cdot \pi/2,$$

which will not be a contraction if $|\lambda| > 2/\pi$. ■

#2

Consider the O.D.E.

$$\dot{x} = x^3$$
$$x(0) = a.$$

Use Picard iterations and show that the iterations converge to the Taylor series.

Solution:

The exact solution satisfies

$$\int_a^x \frac{1}{x^3}\, dx = t$$

$$\Rightarrow -\frac{1}{2x^2} + \frac{1}{2a^2} = t$$

$$\Rightarrow \frac{1}{2a^2} - t = \frac{1}{2x^2}$$

$$\Rightarrow \frac{2a^2}{1-2a^2t} = 2x^2$$

$$\Rightarrow x = \sqrt{\frac{a^2}{1-2a^2t}} = a^2(1-2a^2t)^{-1/2}$$

$$\Rightarrow x = a(1 + a^2t + \tfrac{3}{2}a^4t^2 + \tfrac{5}{2}a^6t^3 + \dots)$$

Using Picard iterations we have

$$x_0 = a$$
$$x_1 = a + \int_0^t x_0^3\, dt = a + a^3t = a(1+a^2t)$$
$$x_2 = a + \int_0^t a^3(1+a^2t)^3\, dt$$
$$= a + \int_0^t a^3(1+3a^2t+3a^4t^2+a^6t^3)\, dt$$
$$= a(1 + a^2t + \tfrac{3}{2}a^4t^2 + a^6t^3 + \tfrac{a^4}{4}t^4)$$ ■

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