MST 750
Homework #6

Due Date: March 18, 2022

1. Consider the the following differential equation
$$\dot{x} = f(x, t),$$
where $f : \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}$ is continuous. Show that if $|f(t, x) - f(t, y)| \le L(t)|x - y|$ then
$$|x(t) - y(t)| \le |x_0 - y_0| \exp\left(\left|\int_{t_0}^{t} L(s)ds\right|\right),$$
where x, y are solutions to the ordinary differential equation satisfying $x(t_0) = x_0$, $y(t_0) = y_0$.
2. Let $u, v, w \in C^0([a, b]; \mathbb{R})$ with $w > 0$ such that
$$u(t) \le v(t) + \int_a^t w(s)u(s)\, ds$$
for every $t \in [a, b]$. Prove that
$$u(t) \le v(t) + \int_a^t w(s)v(s) \exp\left(\int_s^t w(u)\, du\right) ds.$$
3. pg. 153, #1
4. pg. 153, #2
5. pg. 153, #3
6. pg. 153, #4
7. Consider the following differential equation
$$\dot{x} = f(x),$$
where $f : \mathbb{R} \mapsto \mathbb{R}$ is a differentiable function satisfying $f(0) = f(1) = 0$ and $f(x) > 0$ for $x \in (0, 1)$. Determine $\Gamma(x)$ and $\omega(x)$ if $x \in [0, 1]$.
8. Denote by $d(x, A) = \inf_{y \in A} |x - y|$ the distance between a point $x \in \mathbb{R}^n$ and a set $A \subset \mathbb{R}^n$.
   (a) Show that $|d(x, A) - d(z, A)| \le |x - z|$.
   (b) Prove that the mapping $x \mapsto d(x, A)$ is a continuous mapping from $\mathbb{R}^n$ to $\mathbb{R}$.
9. For a function $g \in C^2(\mathbb{R}^2; \mathbb{R})$, consider the equation
$$\dot{x} = -\nabla g(x).$$
   (a) Show that if u is a nonconstant solution, then $g \circ u$ is strictly decreasing.
   (b) Show this system has no periodic orbits.
   (c) For the function $g(x, y) = x^2y^4$ sketch the level sets of $g(x, y)$ overlaid on top of a phase portrait. What geometric condition must hold between the level sets and the orbits?
10. Consider the following equation in polar coordinates:
$$\dot{r} = f(r),$$
$$\dot{\theta} = 1,$$
where
$$f(r) = \begin{cases} r\sin(1/r^2), & r \neq 0 \\ 0, & r = 0 \end{cases}.$$
Show that the origin is Lyapunov stable but not asymptotically stable.